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Partial Differential Equations

Endpoint Strichartz estimate for the kinetic transport equation in one dimension ☆

Zihua Guo, Lizhong Peng

LMAM, School of Mathematical Sciences, Changchun Yuan, Peking University, Beijing 100871, China

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Abstract

In this Note, we consider problems of endpoint Strichartz estimates for the kinetic equation in one dimension. The fundamental result obtained in Theorem 1 is proved using two different methods: in the first we construct an explicit counterexample; in the second uses a duality argument. ***To cite this article: Z. Guo, L. Peng, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Estimations de Strichartz dans un cas limite pour l'équation de transport cinétique unidimensionnelle. Dans cette Note on étudie des problèmes d'estimations de Strichartz dans un cas limite pour l'équation cinétique. Dans le cas de la dimension un, le résultat fondamental du Théorème 1 est démontré par deux méthodes : dans la première on construit un contrexemple explicite, dans le seconde on utilise un argument de dualité. ***Pour citer cet article : Z. Guo, L. Peng, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

We consider Strichartz estimates for the kinetic transport equation,

$$\begin{cases} \frac{\partial}{\partial t} f(t,x,\xi) + \xi \cdot \nabla_x f(t,x,\xi) = 0, \quad (t,x,\xi) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \\ f(0,x,\xi) = f^0(x,\xi). \end{cases} \tag{1}$$

Given (non-negative) f and f_0 as above, we seek all estimates of the form:

$$\|f\|_{L_t^q L_x^p L_\xi^r} \lesssim \|f^0\|_{L^a_{x,\xi}}, \tag{2}$$

where we use the notation $X \lesssim Y$ to denote $X \leqslant CY$ for some constant $C > 0$ independent of f_0. From dimensional analysis, the following conditions are necessary:

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E-mail addresses: zihuaguo@math.pku.edu.cn (Z. Guo), lzpeng@pku.edu.cn (L. Peng).

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$$\frac{2}{q} = n\left(\frac{1}{r} - \frac{1}{p}\right), \qquad \frac{1}{a} = \frac{1}{2}\left(\frac{1}{r} + \frac{1}{p}\right). \tag{3}$$

Also, $p \geqslant a$ and $q \geqslant a$ are necessary (see [2]).

If the above necessary conditions hold and $q > 2 \geqslant a$, (2) was proved in [1]. Then in [2] Keel and Tao weaken the condition to $q > a$. Compared to the necessary condition, the endpoint $q = a$ was left unknown and was conjectured to be true at least when $n > 1$. By simple calculation we know that $f(t, x, \xi) = f^0(x - t\xi, \xi)$ if f_0 is a good function. Hence we can assume that $q = a = 2$, in which case $p = \frac{2n}{n-1}$, $r = \frac{2n}{n+1}$. Therefore, in the endpoint case (2) gives:

$$\|f\|_{L_t^2 L_x^{\frac{2n}{n-1}} L_\xi^{\frac{2n}{n+1}}} \lesssim \|f^0\|_{L^2_{x,\xi}}. \tag{4}$$

In this Note we consider the case $n = 1$, while leaving the case $n > 1$ still open. Our first result is:

Theorem 1. *There does not exist a constant $C > 0$ satisfying, for all $f^0(x, \xi) \in L^2_{x,\xi}$,*

$$\|f\|_{L_t^2 L_x^\infty L_\xi^1} \leqslant C\|f^0\|_{L^2_{x,\xi}}, \tag{5}$$

where $f(t, x, \xi)$ is the solution to (1) *with initial data f^0.*

We will prove Theorem 1 in two different ways. The first method provides an explicit counterexample to (5), while the second one uses a duality argument. A natural extension of the above result is the following: does estimate (5) become true when the L^∞ norm is replaced by the BMO norm? Indeed recall that BMO usually is a substitute for L^∞ in many situations in harmonic analysis. It turns out this is the case in the present situation as well.

More precisely, we have:

Theorem 2. *There exists a constant $C > 0$ such that, for all $f^0(x, \xi) \in L^2_{x,\xi}$,*

$$\|f\|_{L_t^2 BMO_x L_\xi^1} \leqslant C\|f^0\|_{L^2_{x,\xi}}, \tag{6}$$

where $f(t, x, \xi)$ is the solution to (1) *with initial data f^0.*

Theorem 2 is quite surprising at first glance. In the endpoint case for the wave ($n = 3$) or Schrödinger ($n = 2$) equation, we know that Strichartz estimate does not hold even if L^∞ norm replaced by BMO norm (see [3]).

2. Proof of the main results

2.1. Proof of Theorem 1

Let $h(x) = x^{-1/2}(-\log x)^{-\frac{1+\epsilon}{2}}\chi_{\{0<x<1/2\}}$, and $f^0(x, \xi) = h(-x)h(\xi)$, where $0 < \epsilon < 1/2$. We can easily get that $\|h\|_2 < \infty$, hence $\|f^0\|_{L^2_{x,\xi}} < \infty$. However, we have, for $0 < t < 1/2$,

$$\begin{aligned}
\left\|f(t, x, \xi)\right\|_{L_x^\infty L_\xi^1} &\geqslant \left\|f(t, 0, \xi)\right\|_{L_\xi^1} \\
&= \int_0^{1/2} \xi^{-1/2}(-\log \xi)^{-\frac{1+\epsilon}{2}}(t\xi)^{-1/2}(-\log t\xi)^{-\frac{1+\epsilon}{2}}\, \mathrm{d}\xi \\
&\geqslant 2^{-(1+\epsilon)/2} t^{-1/2} \int_0^t \xi^{-1}(-\log \xi)^{-\frac{1+\epsilon}{2}}(-\log \xi)^{-\frac{1+\epsilon}{2}}\, \mathrm{d}\xi \\
&\geqslant ct^{-1/2} \int_0^t \frac{1}{\xi(-\log \xi)^{1+\epsilon}}\, \mathrm{d}\xi \\
&= ct^{-1/2}(-\log t)^{-\epsilon}.
\end{aligned}$$

From the fact that $\|t^{-1/2}(\log t)^{-\epsilon}\|_{L^2_{(0,1/2)}} = \infty$ when $0 < \epsilon < 1/2$, we immediately get that $\|f\|_{L^2_t L^\infty_x L^1_\xi} = \infty$. Therefore, (5) fails.

Next, we give a alternative proof. By duality (5) is equivalent to the following:

$$\left\| \int g(t, x + t\xi, \xi)\,\mathrm{d}t \right\|_{L^2_{x,\xi}} \lesssim \|g\|_{L^2_t L^1_x L^\infty_\xi}. \tag{7}$$

We claim that (7) fails. Let $g(t, x, \xi) = s(t)h(x)$.

$$\begin{aligned}\left\| \int g(t, x + t\xi, \xi)\,\mathrm{d}t \right\|_{L^2_{x,\xi}} &= \left\| \int s(t)h(x + t\xi)\,\mathrm{d}t \right\|_{L^2_{x,\xi}} \\ &= \left\| \hat{s}(-\xi w)\hat{h}(w) \right\|_{L^2_{\xi,w}} \quad \text{(Plancherel equality)} \\ &= \|s\|_{L^2} \left\| |w|^{-1/2}\hat{h}(w) \right\|_{L^2} \\ &= \|s\|_{L^2} \left\| I^{-1/2}h \right\|_{L^2},\end{aligned}$$

where $I^{-1/2}$ is fractional integration operator. Thus (7) reduces to

$$\left\| I^{-1/2}h \right\|_{L^2} \lesssim \|h\|_{L^1}. \tag{8}$$

From the basic fact about fractional integration (see [4]), (8) does not hold for general $h \in L^1$; therefore, (7) fails.

However, (8) is true if the L^1 norm on the right side is replaced by the H^1 (Hardy space) norm (see [4]). Thus we ask that whether (7) is true if we replace L^1 norm on the right side by H^1 norm. The answer is positive.

2.2. *Proof of Theorem 2*

By duality (6) is equivalent to the following:

$$\left\| \int g(t, x + t\xi, \xi)\,\mathrm{d}t \right\|_{L^2_{x,\xi}} \lesssim \|g\|_{L^2_t H^1_x L^\infty_\xi}. \tag{9}$$

First we prove that for all nonnegative $g(t, x) \in L^2_t H^1_x$, we have:

$$\left\| \int g(t, x + t\xi)\,\mathrm{d}t \right\|_{L^2_{x,\xi}} \lesssim \|g\|_{L^2_t H^1_x}. \tag{10}$$

From Plancherel equality and boundedness of $I^{-1/2} : H^1 \to L^2$, use $\mathfrak{F}_2$ to denote the Fourier transform with respect to the second variable,

$$\begin{aligned}\left\| \int g(t, x + t\xi)\,\mathrm{d}t \right\|_{L^2_{x,\xi}} &= \left\| \int g(t, x + t\xi)\,\mathrm{d}t \right\|_{L^2_{x,\xi}} \\ &= \left\| \hat{g}(-\xi w, w) \right\|_{L^2_{\xi,w}} \\ &= \left\| |w|^{-1/2}\mathfrak{F}_2 g(t, w) \right\|_{L^2_{t,w}} \\ &= \left\| I^{-1/2}\big(g(t, \cdot)\big)(w) \right\|_{L^2_{t,w}} \\ &\lesssim \|g\|_{L^2_t H^1_x}.\end{aligned}$$

For general g, let $f(t, x) = \|g(t, x, \xi)\|_{L^\infty_\xi}$. From (10), we have:

$$\left\| \int g(t, x + t\xi, \xi)\,\mathrm{d}t \right\|_{L^2_{x,\xi}} \leqslant \left\| \int f(t, x + t\xi)\,\mathrm{d}t \right\|_{L^2_{x,\xi}} \lesssim \|g\|_{L^2_t H^1_x L^\infty_\xi},$$

which completes the proof of (9).

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