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Differential Topology/Dynamical Systems

Asymptotic Rasmussen invariant

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Abstract

We use simple properties of the Rasmussen invariant of knots to study its asymptotic behaviour on the orbits of a smooth volume preserving vector field on a compact domain of the 3-space. A comparison with the asymptotic signature allows us to prove that asymptotic knots of non-zero invariant are non-alternating. ***To cite this article: S. Baader, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Invariant de Rasmussen asymptotique. Dans cette Note, nous étudions l'invariant de Rasmussen des nœuds asymptotiques d'un champ de vecteurs préservant une mesure de probabilité sur un domaine compact de l'espace. Plus précisément, nous démontrons que l'invariant de Rasmussen des nœuds asymptotiques est égal au double de la signature asymptotique. Par conséquent, les nœuds asymptotiques d'invariant non-nul sont non-alternés. La preuve présentée ici est inspirée de la technique de Gambaudo et Ghys. ***Pour citer cet article : S. Baader, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

The helicity of a smooth volume-preserving vector field on a closed homology 3-sphere measures how pairs of orbits are linked asymptotically, in the average (see [1]). An asymptotic linking number for pairs of orbits does also exist for a smooth volume-preserving vector field X on a compact domain $G \subset \mathbb{R}^3$ with smooth boundary, provided X is tangent to the boundary ∂G. In [2], Gambaudo and Ghys proved that the asymptotic linking number can be determined by looking at a single orbit only. More precisely, they proved the existence of an asymptotic signature invariant that coincides with the helicity of X divided by two, for almost all orbits, provided the flow of X is ergodic.

In this Note we prove the existence of an asymptotic Rasmussen invariant, answering thereby a question posed by Ghys in [3]. Comparing this invariant with the asymptotic signature, we show that asymptotic knots are non-alternating, as soon as the helicity of X is non-zero. In order to state our main theorem, we have to describe how pieces of orbits can be turned into knots: let $x \in G$ be a non-periodic, non-singular point for the flow Φ_X of the vector field X. For a fixed time $T > 0$, we define $K(T, x) \subset \mathbb{R}^3$ to be the piece of orbit from x to $\Phi_X(x, T)$, followed by the geodesic segment γ joining $\Phi_X(x, T)$ to x (γ need not be contained in G). A careful exposition on the closure

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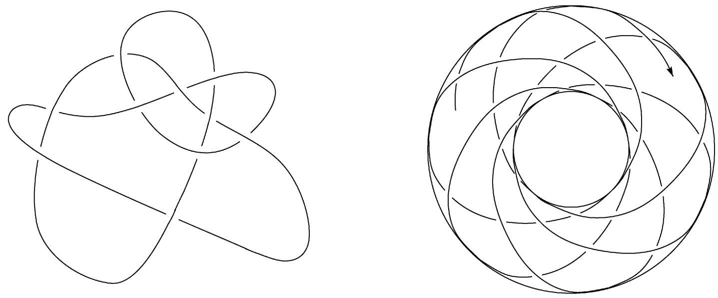

Fig. 1. Alternating knot diagram and non-alternating piece of orbit.

of pieces of orbits can be found in [8]. For almost all $x \in G$ and $T > 0$, $K(T,x)$ is an embedded curve, i.e. a knot. As usual, we denote the Rasmussen invariant and the signature of a knot K by $s(K)$ and $\sigma(K)$, respectively. We shall review some basic properties of the Rasmussen invariant at the beginning of the next section.

Theorem 1.1. *Let X be a smooth vector field on a compact domain $G \subset \mathbb{R}^3$, tangent to the boundary ∂G, with hyperbolic singularities only, i.e. linear singularities corresponding to critical points of index 1 or 2 of a Morse function on $\mathbb{R}^3$. If μ is an X-invariant probability measure which does not charge the periodic orbits and singular points of X, then the limit*

$$s(X,x) := \lim_{T\to\infty} \frac{1}{T^2} s\big(K(T,x)\big)$$

exists for almost all $x \in G$ (with respect to μ) and coincides with

$$2\sigma(X,x) := 2 \lim_{T\to\infty} \frac{1}{T^2} \sigma\big(K(T,x)\big).$$

Remark 1. As Gambaudo and Ghys observed in [2], the quantity $2\sigma(X,x)$ coincides with the helicity (or Arnold invariant) of X, for almost all $x \in G$, if the flow of X is ergodic with respect to μ, i.e. if every measurable function which is invariant under the flow of X is constant almost everywhere.

Theorem 1.1 allows us to detect the non-alternating character of long pieces of orbits, in general. A knot or link is called *alternating*, if it has an alternating planar diagram, i.e. a diagram in which we encounter the crossings in an alternating way on an upper strand and a lower strand, as shown on the left-hand side of Fig. 1.

Corollary 1.2. *If the flow of X is ergodic with respect to μ, and if, in addition, X has non-zero helicity, then for almost all points $x \in G$, there exists a positive constant $S \in \mathbb{R}$ such that the knot $K(T,x)$ is non-alternating, for almost all $T \geqslant S$.*

As an example, let X be the constant vector field $(1,\omega)$ on $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$. We can easily extend X to a non-vanishing vector field on the full torus $V = S^1 \times D^2$, which we may view as a domain in $\mathbb{R}^3$. If we choose an irrational slope $\omega \in \mathbb{R} - \mathbb{Q}$, then the orbit starting at a point $x \in S^1 \times S^1$ is non-periodic and shows a typical non-alternating behaviour, as shown on the right-hand side of Fig. 1. Its asymptotic Rasmussen invariant is $s(X,x) = 2\sigma(X,x) = \frac{1}{4\pi^2}\omega$ (see [2], p. 50).

Remark 2. We do not know whether the assumption of Corollary 1.2 that the helicity be non-zero is essential. However, we cannot drop both assumptions, as shows the example of the constant vector field $(1,0)$ on $S^1 \times D^2 = \mathbb{R}/\mathbb{Z} \times D^2$, whose orbits are all periodic and unknotted.

The proof of Theorem 1.1 is based upon the notion of 'good' diagrams for long pieces of orbits, whose existence was proved by Gambaudo and Ghys in [2]. We present it in the next section, together with the proof of Corollary 1.2.

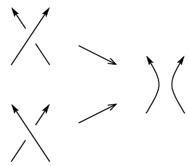

Fig. 2. Smoothing of crossings.

2. The asymptotic Rasmussen invariant

The Rasmussen invariant s of knots was constructed from the Khovanov complex of knots in [5]. Among various interesting properties of the Rasmussen invariant, two are of special interest to us. The first one is a Bennequin-style inequality:

$$s(K) \geqslant 1 + w(D) - o(D), \tag{1}$$

where $w(D)$ and $o(D)$ stand for the writhe and the number of Seifert circles of a diagram D of K, respectively. The *writhe* of a knot or link diagram, also known as the algebraic crossing number, is the number of positive crossings minus the number of negative crossings of that diagram. The *Seifert circles* are those embedded circles in the plane that arise from smoothing all the crossings of a knot or link diagram. Fig. 2 shows how to smooth a positive or negative crossing of a diagram.

The second important property is an equality that relates the Rasmussen invariant and the signature of alternating knots:

$$s(K) = \sigma(K), \tag{2}$$

for all alternating knots K.

The first inequality was proved by Shumakovitch in [7], based upon a general principal due to Rudolph [6], the other property appears in Rasmussen's original paper. Applying the first inequality to the mirror image $\bar{K}$ of a knot K, we obtain

$$s(\bar{K}) \geqslant 1 + w(\bar{D}) - o(\bar{D}) = 1 - w(D) - o(D).$$

Combining this with the fact that $s(\bar{K}) = -s(K)$, we also get an upper bound for $s(K)$, altogether:

$$1 + w(D) - o(D) \leqslant s(K) \leqslant -1 + w(D) + o(D). \tag{3}$$

According to Gambaudo and Ghys [2], under the assumptions of Theorem 1.1, the complement of the singularities of X can be covered by an enumerable family of flow boxes whose flow time (i.e. the minimal time it takes to pass through a flow box) is bounded from below by a global constant $\lambda > 0$. Further they show that for almost all $x \in G$ and $T > 0$, the knots $K(T, x)$ have diagrams $\pi_0(K(T, x))$, coming from a good projection π_0 onto a plane, whose writhe, called θ there, grows quadratic in T, more precisely:

$$\lim_{T\to\infty} \frac{1}{T^2} w\bigl(\pi_0\bigl(K(T, x)\bigr)\bigr) = 2\sigma(X, x). \tag{4}$$

In fact, they subdivide the crossings of $\pi_0(K(T, x))$ into three types ([2], p. 64): D_1, D_2, D_3. The crossings of type D_1 arise from overcrossing flow boxes, as illustrated in Fig. 3, on the left-hand side; their number grows quadratic in T. The number of crossings of types D_2 and D_3 grows subquadratic in T.

We shall estimate the number of Seifert circles $o(\pi_0(K(T, x)))$. Every Seifert circle of $\pi_0(K(T, x))$ is adjacent to at least one crossing, and there are at most two Seifert circles meeting at each crossing (compare Fig. 2). Therefore the number of Seifert circles adjacent to a crossing of type D_2 or D_3 grows subquadratic in T. Further, every Seifert circle which is adjacent to a crossing of type D_1 must enter at least one flow box (see Fig. 3, on the right-hand side). Therefore there are at most $\frac{T}{\lambda}$ such circles, where λ is the minimal flow time for flow boxes! Altogether, this shows

$$\lim_{T\to\infty} \frac{1}{T^2} o\bigl(\pi_0\bigl(K(T, x)\bigr)\bigr) = 0.$$

In view of (3) and (4), this proves Theorem 1.1.

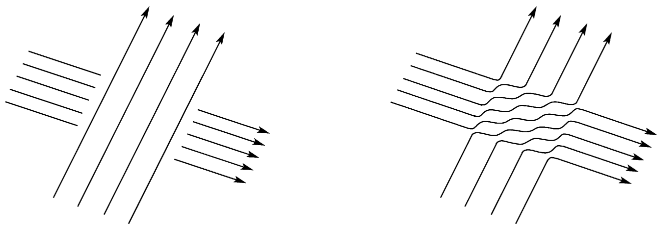

Fig. 3. Two overcrossing flow boxes.

Corollary 1.2 in turn is an immediate consequence of Theorem 1.1 and the fact that $s(K) = \sigma(K)$, for all alternating knots K. Indeed, if the vector field X is ergodic, then its helicity coincides with $2\sigma(X, x)$, for almost all $x \in G$ [2]. If, in addition, the helicity is non-zero, then $2\sigma(X, x) \neq 0$ and $s(X, x) = 2\sigma(X, x) \neq \sigma(X, x)$, for almost all $x \in G$, hence there exists a constant $S \geqslant 0$, such that $s(K(T, x)) \neq \sigma(K(T, x))$, for almost all $T \geqslant S$.

Remark 3. The proof of Theorem 1.1 works for all knot invariants I that satisfy the inequality $I(K) \geqslant 1 + w(D) - o(D)$ and the equation $I(\bar{K}) = -I(K)$. This is notably the case for the invariant 2τ coming from knot Floer homology [4].

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