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Combinatorics

The 2-color relative linear Van der Waerden numbers ☆

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Abstract

We define the r-color relative linear van der Waerden numbers for a positive integer r as generalizations of the polynomial van der Waerden numbers of linear polynomials. Especially we express a sharp upper bound of the 2-color relative linear van der Waerden number $Rf_2(u_1, u_2, \ldots, u_m : s_1, s_2, \ldots, s_k)$ in terms of a $(k+1)$-color polynomial van der Waerden number for positive integers $m, k, u_1, u_2, \ldots, u_m, s_1, s_2, \ldots, s_k$. As a result, we find this upper bound for some instances of $m, k, u_1, u_2, \ldots, u_m, s_1, s_2, \ldots, s_k$ for which the $(k+1)$-color polynomial van der Waerden numbers are obtained. ***To cite this article: B.M. Kim, Y. Rho, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Les nombres de van der Waerden linéaires relatifs 2-colorés. Nous définissons les nombres de van der Waerden linéaires relatifs r-colorés pour un entier strictement positif r qui sont des généralisations des nombres polynomiaux de van der Waerden de polynôme linéaires. En particulier nous donnons, pour $r = 2$, la borne supérieure de ces nombres $Rf_2(u_1, u_2, \ldots, u_m : s_1, s_2, \ldots, s_k)$ en termes d'un nombre de van der Waerden polynomial $(k+1)$-coloré pour les entiers strictement positifs, $m, k, u_1, u_2, \ldots, u_m, s_1, s_2, \ldots, s_k$. Comme conséquence, nous obtenons explicitement cette borne supérieure pour certaines valeurs de ces entiers pour lesquels les nombres polynomiaux de van der Waerden $(k+1)$-colorés peuvent être calculés. ***Pour citer cet article : B.M. Kim, Y. Rho, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

Throughout this Note, we denote $\{1, 2, \ldots, n\}$ by $[n]$ for a positive integer n. Let r be a positive integer. The van der Waerden theorem [6] which states that for every positive integer k, there is a smallest positive integer $w(k)$ such that every r-coloring of $[w(k)]$ has a monochromatic k-term arithmetic progression is a classical result of Ramsey theory on the positive integers; see [4]. This theorem was generalized to the polynomial van der Waerden theorem by Bergelson and Leibman [2]. They proved that for polynomials with rational coefficients $p_1(y), p_2(y), \ldots, p_k(y)$ which take integer values on the positive integers with $p_1(0) = p_2(0) = \cdots = p_k(0) = 0$, there is a smallest positive

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integer w which satisfies that for all r-coloring of $[w]$, $x_0, x_0+p_1(y_0), x_0+p_2(y_0), \ldots, x_0+p_k(y_0) \in [w]$ and $x_0, x_0+p_1(y_0), x_0+p_2(y_0), \ldots, x_0+p_k(y_0)$ are monochromatic for some $x_0 \in [w]$ and $y_0 \in Z^+$. They called w the r-color polynomial van der Waerden number $V_r(p_1(y), p_2(y), \ldots, p_k(y))$. Brown, Landman and Mishna [3] considered the r-color polynomial van der Waerden numbers where all the polynomials are of degree 1 with integral coefficients. For positive integers $a_1, a_2, \ldots, a_k$, they denoted $V_r(a_1 y, (a_1+a_2)y, \ldots, (a_1+\cdots+a_k)y)$ by $f^{(r)}(a_1, a_2, \ldots, a_k)$. In particular, $f^{(2)}(a_1, a_2)$ has been found by Brown, Landman and Mishna [3] in many cases and by the authors of this paper [5] in the remaining cases.

For each $1 \in [r]$, let k_i be positive integers and $\{a_{ij}\}$ be a set of positive integers for $j \in [k_i]$. Define the r-color relative linear van der Waerden number

$$Rf_r(a_{11}, a_{12}, \ldots, a_{1k_1} : a_{21}, a_{22}, \ldots, a_{2k_2} : \cdots : a_{r1}, a_{r2}, \ldots, a_{rk_r})$$

as the smallest positive integer w such that for every r-coloring $C : [w] \to \{0, 1, \ldots, r-1\}$, there is $i \in [r]$ which satisfies that $x + (a_{i1} + a_{i2} + \cdots + a_{ik_i})y \in [w]$ and

$$C(x) = C(x + a_{i1}y) = C\big(x + (a_{i1} + a_{i2})y\big) = \cdots = C\big(x + (a_{i1} + a_{i2} + \cdots + a_{ik_i})y\big) = i - 1$$

for some $x \in [w]$ and $y \in \mathbb{Z}^+$. Note that

$$Rf_r(a_1, a_2, \ldots, a_k : a_1, a_2, \ldots, a_k : \cdots : a_1, a_2, \ldots, a_k) = f^{(r)}(a_1, a_2, \ldots, a_k)$$

for positive integers $a_1, a_2, \ldots, a_k$ and

$$\begin{aligned} &Rf_r(a_{11}, a_{12}, \ldots, a_{1k_1} : a_{21}, a_{22}, \ldots, a_{2k_2} : \cdots : a_{r1}, a_{r2}, \ldots, a_{rk_r}) \\ &\quad = Rf_r(a_{\pi(1)1}, a_{\pi(1)2}, \ldots, a_{\pi(1)k_{\pi(1)}} : a_{\pi(2)1}, a_{\pi(2)2}, \ldots, a_{\pi(2)k_{\pi(2)}} : \cdots : a_{\pi(r)1}, a_{\pi(r)2}, \ldots, a_{\pi(r)k_{\pi(r)}}) \end{aligned}$$

for a permutation π of $\{1, 2, \ldots, r\}$. In this Note, we prove that $f^{(k+1)}(u_1, u_2, \ldots, u_m) + \sum_{i=1}^k s_i$ is a sharp upper bound of the 2-color relative linear van der Waerden number $Rf_2(u_1, u_2, \ldots, u_m : s_1, s_2, \ldots, s_k)$ and so is $f^{(m+1)}(s_1, s_2, \ldots, s_k) + \sum_{i=1}^m u_i$.

2. Results

Theorem 2.1. *Let $u_1, u_2, \ldots, u_m, s_1, s_2, \ldots, s_k \in \mathbb{Z}^+$. Also let $M = \max\{u_1, \frac{u_r}{u_{r-1}} \mid 2 \leqslant r \leqslant m\}$ and $\alpha = f^{(k+1)}(u_1, u_2, \ldots, u_m)$. Then*

$$Rf_2(u_1, u_2, \ldots, u_m : s_1, s_2, \ldots, s_k) \leqslant \alpha + \sum_{i=1}^k s_i. \tag{1}$$

The equality in (1) *holds if*

$$s_1 > M(\alpha - 2) + \alpha - 2$$

and

$$s_j > M\left(\sum_{i=1}^{j-1} s_i + \alpha - 2\right) + \alpha - 2$$

for all $j = 2, 3, \ldots, k$.

Proof. Let $\beta = \alpha + \sum_{i=1}^k s_i$. To show inequality (1), we need to show that every coloring $C : [\beta] \to \{0, 1\}$ satisfies that for some $x \in [\beta]$, $y \in \mathbb{Z}^+$, $x + (\sum_{i=1}^m u_i)y \in [\beta]$ and $C(x) = C(x + (\sum_{i=1}^l u_i)y) = 0$ for all $l \in [m]$, or for some $x \in [\beta]$, $y \in \mathbb{Z}^+$, $x + (\sum_{i=1}^k s_i)y \in [\beta]$ and $C(x) = C(x + (\sum_{i=1}^j s_i)y) = 1$ for all $j \in [k]$. Let C be a coloring $C : [\beta] \to \{0, 1\}$ which does not satisfy the latter. Then we need to show that C satisfies the former. Define $D : [\alpha] \to \{0, 1, \ldots, k\}$ by

$$D(x) = \begin{cases} 0 & \text{if } C(x) = 0, \\ \text{the smallest } j \in [k] \text{ such that } C(x + \sum_{i=1}^j s_i) = 0 & \text{otherwise.} \end{cases}$$

Then D is well-defined when for all $x \in [\alpha]$, $C(x) = 0$ or $C(x + \sum_{i=1}^{j} s_i) = 0$ for some $j \in [k]$. While when D is a $(k+1)$-coloring of $[\alpha]$, there are $x \in [\alpha]$ and $y \in \mathbb{Z}^+$ such that $x + (\sum_{i=1}^{m} u_i)y \in [\alpha]$ and $D(x) = D(x + (\sum_{i=1}^{l} u_i)y)$ for all $l \in [m]$. If $D(x) = 0$, then $C(x) = C(x + (\sum_{i=1}^{l} u_i)y) = 0$ for all $l \in [m]$. If $D(x) = j$ for some $j \in [k]$, then $x + \sum_{i=1}^{j} s_i + (\sum_{i=1}^{m} u_i)y \in [\beta]$ and $C(x + \sum_{i=1}^{j} s_i) = C(x + \sum_{i=1}^{j} s_i + (\sum_{i=1}^{l} u_i)y) = 0$ for all $l \in [m]$. Therefore, in any case, C satisfies the former. Thus (1) is true.

Assume that $s_1 > M(\alpha - 2) + \alpha - 2$ and $s_j > M(\sum_{i=1}^{j-1} s_i + \alpha - 2) + \alpha - 2$ for all $j = 2, 3, \ldots, k$. To prove the condition for (1) to be an equality, it is enough to find a coloring $C_1 : [\beta - 1] \to \{0, 1\}$ which satisfies that for no $x \in [\beta - 1]$, $y \in \mathbb{Z}^+$, $x + (\sum_{i=1}^{m} u_i)y \in [\beta - 1]$ and $C_1(x) = C_1(x + (\sum_{i=1}^{l} u_i)y) = 0$ for all $l \in [m]$ and for no $x \in [\beta - 1]$, $y \in \mathbb{Z}^+$, $x + (\sum_{i=1}^{k} s_i)y \in [\beta - 1]$ and $C_1(x) = C_1(x + (\sum_{i=1}^{j} s_i)y) = 1$ for all $j \in [k]$. As $\alpha = f^{(k+1)}(u_1, u_2, \ldots, u_m)$, there is a coloring $D_1 : [\alpha - 1] \to [0, k]$ such that for no $x \in [\alpha - 1]$ and $y \in \mathbb{Z}^+$, $x + (\sum_{i=1}^{m} u_i)y \in [\alpha - 1]$ and $D_1(x) = D_1(x + (\sum_{i=1}^{l} u_i)y)$ for all $l \in [m]$. Define $C_1 : [\beta - 1] \to \{0, 1\}$ by

$$C_1(x) = \begin{cases} 0 & \text{if either } D_1(x) = 0 \text{ or } x = z + (\sum_{i=1}^{j} s_i) \text{ for some } z \in [\alpha - 1] \\ & \text{and } j \in [k] \text{ such that } D_1(z) = j, \\ 1 & \text{otherwise.} \end{cases}$$

If C_1 does not satisfy the latter and hence there are $x \in [\beta - 1]$, $y \in \mathbb{Z}^+$ such that $x + (\sum_{i=1}^{k} s_i)y \in [\beta - 1]$ and $C_1(x) = C_1(x + (\sum_{i=1}^{j} s_i)y) = 1$ for all $j \in [k]$, then $x \in [\alpha - 1]$ and $y = 1$. $D_1(x) \neq 0$ since $C_1(x) = 1$. Thus $D_1(x) = j$ for some $j \in [k]$ and hence $C_1(x + \sum_{i=1}^{j} s_i) = 0$. This gives a contradiction.

Suppose C_1 does not satisfy the former and hence there are $x \in [\beta - 1]$, $y \in \mathbb{Z}^+$ such that $x + (\sum_{i=1}^{m} u_i)y \in [\beta - 1]$ and $C_1(x) = C_1(x + (\sum_{i=1}^{l} u_i)y) = 0$ for all $l \in [m]$. Let

$$S_j = \begin{cases} [\alpha - 1] & \text{if } j = 0, \\ [\sum_{i=1}^{j} s_i + 1, \sum_{i=1}^{j} s_i + \alpha - 1] & \text{if } j \in [k]. \end{cases}$$

As $C_1(x) = 0$, either $D_1(x) = 0$ or $x = z + (\sum_{i=1}^{j} s_i)$ for some $z \in [\alpha - 1]$ and hence either $x \in S_0$ or $x \in S_j$ for some $j \in [k]$. Thus $x \in S_{p_0}$ for some $p_0 \in [0, k]$. Similarly, $x + (\sum_{i=1}^{l} u_i)y \in S_{p_l}$ for some $p_l \in [0, k]$ for all $l \in [m]$. As $x < x + u_1 y < \cdots < x + (u_1 + \cdots + u_m)y$, $p_0 \leqslant \cdots \leqslant p_m$. Let $p = p_0$ and $q = p_m$. Consider the case where $p \geqslant 1$ first. Suppose that $p = q$. Then $x, x + (\sum_{i=1}^{l} u_i)y \in S_p$ for all $l \in [m]$. Let $x_0 = x - \sum_{i=1}^{p} s_i$. Then $x_0, x_0 + (\sum_{i=1}^{l} u_i)y \in S_0$ and $D_1(x_0) = D_1(x_0 + (\sum_{i=1}^{l} u_i)y) = p$ for all $l \in [m]$. This gives a contradiction. Therefore $p < q$. Then there is a smallest $r \in [m]$ such that $x + (\sum_{i=1}^{r} u_i)y \in S_q$. Firstly assume that $r = 1$. Then as $x \in S_p$ and $x + u_1 y \in S_q$,

$$x + u_1 y - x \geqslant \sum_{i=1}^{q} s_i + 1 - \left(\sum_{i=1}^{p} s_i + \alpha - 1\right) \geqslant s_q - \alpha + 2 > M\left(\sum_{i=1}^{q-1} s_i + \alpha - 2\right) \geqslant u_1\left(\sum_{i=1}^{q-1} s_i + \alpha - 2\right)$$

and hence

$$y > \sum_{i=1}^{q-1} s_i + \alpha - 2.$$

Also as both $x + u_1 y, x + (u_1 + u_2)y \in S_q$,

$$x + (u_1 + u_2)y - (x + u_1 y) \leqslant \alpha - 2$$

and hence

$$u_2 y \leqslant \alpha - 2.$$

Thus we get a contradiction. Secondly assume that $r \geqslant 2$. Then as $x, x + (\sum_{i=1}^{r-1} u_i)y \notin S_q$,

$$x + \left(\sum_{i=1}^{r-1} u_i\right)y - x \leqslant \sum_{i=1}^{q-1} s_i + \alpha - 2$$

and hence

$$\left(\sum_{i=1}^{r-1} u_i\right) y \leqslant \sum_{i=1}^{q-1} s_i + \alpha - 2.$$

Also as $x + (\sum_{i=1}^{r-1} u_i)y \notin S_q$ and $x + (\sum_{i=1}^{r} u_i)y \in S_q$,

$$x + \left(\sum_{i=1}^{r} u_i\right) y - \left(x + \left(\sum_{i=1}^{r-1} u_i\right) y\right) \geqslant \sum_{i=1}^{q} s_i + 1 - \left(\sum_{i=1}^{q-1} s_i + \alpha - 1\right)$$
$$= s_q - \alpha + 2 > M\left(\sum_{i=1}^{q-1} s_i + \alpha - 2\right) \geqslant \frac{u_r}{u_{r-1}}\left(\sum_{i=1}^{q-1} s_i + \alpha - 2\right)$$

and hence

$$u_{r-1} y > \sum_{i=1}^{q-1} s_i + \alpha - 2.$$

Thus we get a contradiction also. In the case where $p = 0$, we get contradictions similarly. Thus inequality (1) is an equality. □

Corollary 2.2.

(i) *For* $s_1, s_2 \in \mathbb{Z}^+$, $Rf_2(1, 1 : s_1, s_2) \leqslant s_1 + s_2 + 27$ *where the equality holds if* $s_1 > 50$ *and* $s_2 > s_1 + 50$.
(ii) *For* $s_1, s_2, s_3 \in \mathbb{Z}^+$, $Rf_2(1, 1 : s_1, s_2, s_3) \leqslant s_1 + s_2 + s_3 + 76$ *where the equality holds if* $s_1 > 148$, $s_2 > s_1 + 148$ *and* $s_3 > s_1 + s_2 + 148$.
(iii) *For* $s_1, s_2 \in \mathbb{Z}^+$, $Rf_2(1, 2 : s_1, s_2) \leqslant s_1 + s_2 + 42$ *where the equality holds if* $s_1 > 120$ *and* $s_2 > 2s_1 + 120$.
(iv) *For* $s_1, s_2 \in \mathbb{Z}^+$, $Rf_2(1, 3 : s_1, s_2) \leqslant s_1 + s_2 + 57$ *where the equality holds if* $s_1 \geqslant 220$ *and* $s_2 > 3s_1 + 220$.

Proof. (i), (ii): The polynomial van der Waerden numbers $f^{(3)}(1, 1) = 27$ and $f^{(4)}(1, 1) = 76$ are known. (See [1].) (iii), (iv): We made exhaustive searches following a brute-force algorithm to verify that the polynomial van der Waerden numbers are given by $f^{(3)}(1, 2) = 42$ and $f^{(3)}(1, 3) = 57$. □

Remark 1. In fact by making an exhaustive search following a brute-force algorithm as in the proof of Corollary 2.2, for any positive integers m, k, $u_1, u_2, \ldots, u_m$, $s_1, s_2, \ldots, s_k$, we can find the polynomial van der Waerden number $f^{(k+1)}(u_1, u_2, \ldots, u_m)$ and hence we can find the upper bound of $Rf_2(u_1, u_2, \ldots, u_m : s_1, s_2, \ldots, s_k)$ which is stated in Theorem 2.1.

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