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Differential Geometry

The Mass according to Arnowitt, Deser and Misner

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Abstract

For asymptotically Euclidean manifolds of order $\tau > (n-2)/2$, under the hypothesis that the mass m (according to Arnowitt, Deser and Misner) exists (in particular if the scalar curvature is $\geqslant 0$ and integrable), there exists a real number $A > 0$ such that $m \geqslant 4(n-1)A$ on each end (except if the metric is Euclidean). ***To cite this article: T. Aubin, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Le Masse selon Arnowitz, Deser et Misner. Pour une variété asymptotiquement euclidienne d'ordre $\tau > (n-2)/2$, sous l'hypothèse que la masse m (selon Arnowitt, Deser et Misner) existe (notamment si la courbure scalaire est $\geqslant 0$ et intégrable), il existe un réel $A > 0$ tel que $m > 4(n-1)A$ sur chaque bout (sauf si la métrique est euclidienne). ***Pour citer cet article : T. Aubin, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Pour une variété asymptotiquement euclidienne d'ordre $\tau > (n-2)/2$, on suppose que la masse m est définie. Cela a lieu en particulier si la courbure scalaire est $\geqslant 0$ et intégrable (hypothèses généralement adoptée mais non nécéssaire). Par une inversion sur chaque bout, après un changement de métrique conforme, le problème est ramené à l'étude de la masse A en un point P d' une variété V compacte dont le Laplacien conforme L est inversible. Ce problème a été traité dans [5] dans toute sa généralité. L'hypothèse $\tau > (n-2)/2$ correspond à $2\omega > n-6$ qui est une condition nécessaire et suffisante pour que la masse $A(P)$ soit > 0 (sauf sur la sphère). On en déduit que, pour ce bout (envoyé sur un voisinage de $P \in V$), la masse $m \geqslant 4(n-1)A$ est positive.

1. Introduction

Let $(V_n, \hat{g})$ be a Riemannian compact C^∞ manifold. According to J. Cao [8] or M. Günter [9], there exists a metric $g \in [\hat{g}]$ conformal to $\hat{g}$ such that $|g| = 1$ in a ball $B_P(\delta)$ in a geodesic coordinates system (r, θ) at P. We define ω

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as the greatest integer for which $\|\nabla^l W(P)\| = 0$ when $|l| < \omega$, W being the Weyl tensor and l a multi-index with length $|l|$.

Denote by $G(P, Q)$ the Green function of L, proportional to $\Psi(P, .)$ defined by:

$$L\Psi = \Delta\Psi + (n-2)R\Psi/4(n-1) = \delta_P,$$

which has r^{2-n} as its principal part at P. Here R is the scalar curvature, δ_P the Dirac measure at P and $r = d(P, Q)$. In [5], see also [6], we proved, in all generality, the Positive Mass Conjecture for compact manifolds when L is invertible. In the ball $B_P(\delta)$, define the Mass A at P by ($\overline{\int}$ means mean value)

$$A = \lim_{r \to 0} \overline{\int_{S_{n-1}(1)}} H(r, \theta)\, \mathrm{d}\sigma(\theta).$$

where $H(P, Q) = G(P, Q) - r^{2-n}$.

Theorem 1. *Let $P \in (V_n, g)$ be a compact manifold with L invertible. The necessary and sufficient condition in order that the mass A at P be finite is $2\omega(P) > n - 6$. Then $A(P) > 0$, except for the sphere in which case $A = 0$.*

So far, the common hypothesis was $H(P, Q) = A + 0(r)$. Now $H(P, Q)$ may have singularities at P. The hypothesis $2\omega(P) > n - 6$, implies that A is finite. Then we proceed as when $H(P, Q)$ was bounded. In this case, it is easier to consider (at the top of p. 82 of [5]) a radial function γ (with $\gamma/\partial W = 1$) instead of the function f.

With $\psi(r, \theta) = \delta^{n-2}G(r, \theta)/(1 + A\delta^{n-2})$, we get, for some $\beta \geqslant 1$,

$$J(\psi) \leqslant J(\tilde{f}) = j(\gamma) + \mathrm{O}\big(\delta^{n-2+\beta}\big).$$

Remember that in $B(\delta_0)$, g is the Cao–Günter's metric ([8] or [9]). For some $k > 0$ when $r < 2r_o$ and $k = 0$ when $r > 2r_o$, we have

$$\overline{\int_{S_{n-1}(1)}} R(r, \theta)\, \mathrm{d}\sigma(\theta) < kr^{\alpha}$$

for some $\alpha > n - 4$. Moreover, the injectivity radius at P is as large as one wants. Nothing is changed, using the fact that $2\omega > n - 6$ implies that the mean value of $H(P, Q)$ on $S_{n-1}(1)$ is finite.

We introduce the change of variable

$$r \to \rho = \left(\delta^{2-n} - (n-2)\int_{\delta}^{r} r^{1-n}\, \mathrm{d}r/h(r)\right)^{-1/(n-2)}$$

(where $h(r)$ is as in [5]), and the function

$$\gamma(\rho) = \sup\big\{0, \big[(\delta/\rho)^{n-2} - \epsilon\delta^{n-2}\big]/\big(1 - \epsilon\delta^{n-2}\big)\big\}.$$

In Physics, Arnowitt, Deser and Misner [1,2] defined the 'ADM mass' for an isolated gravitational system of dimension 3 of space. Witten proved in [16] the positiveness of the mass when $n = 3$, using the spinors, an idea used again by Parker–Taubes [12] and extended to the spin-manifolds of any dimension by Lee–Parker [10]. Recently, a paper of Lohkamp [11] appeared about this topic. We will comment elsewhere about the works of Schoen and Schoen–Yau [13–15].

2. Mass

Let us consider the metric $\tilde{g} = G^{4/(n-2)}g$ (see [3]) on a manifold $W = V\backslash\{P\}$ and let $\rho = 1/r$. Set $K = V\backslash B_P(\delta)$, where $\delta > 0$ is small as usual.

The manifold $(W, \tilde{g})$ is complete, asymptotically Euclidean. Its scalar curvature $\tilde{R}$ vanishes identically and $\Omega = W\backslash K$ is diffeomorphic to $B_P(\delta)\backslash\{P\}$ by the inverse mapping $\rho \to r$.

According to our study in [4–6], $g = \mathcal{E} + \mathrm{O}(r^{2+\omega})$ (where $\mathcal{E}$ denotes the Euclidean metric). It follows that $\tilde{g} = \mathcal{E} + \mathrm{O}(\rho^{-(2+\omega)}) + \mathrm{O}(\rho^{2-n})$.

Remark 1. If we had started from a conformal metric $\bar{g} = \varphi^{4/(n-2)} g$ (if we want $\varphi(P) = 1$), as $\bar{G}(P, Q) = G(P, Q)/\varphi(Q)\varphi(P)$, we would obtain on W the same metric $\tilde{g}$.

Definition 1. The mass $m(\tilde{g})$ of the asymptotically flat manifold $(W, \tilde{g})$ is the limit of

$$\gamma(\rho) = \omega_{n-1}^{-1} \int_{S_{n-1}(1)} \sqrt{\left|\tilde{g}(\rho,\theta)\right|}\, \tilde{g}^{ij} (\partial_i \tilde{g}_{\rho j} - \partial_\rho \tilde{g}_{ij})(\rho,\theta)\rho^{n-1}\, \mathrm{d}\sigma(\theta)$$

when $\rho \to \infty$ if it exists (here $\rho > 1/\delta$).

We showed in [5] that the mass makes sense (is finite) if $2\omega > n - 6$. With the coordinate system that we have defined there, letting $\psi = (r^{n-2} G)^{2/(n-2)}$, shows that this expression is equal to:

$$\gamma(\rho) = \omega_{n-1}^{-1} \int_{S_{n-1}(1)} \left[\psi^n \partial_\rho \psi^2 - \partial_\rho \psi^n\right] \rho^{n-1}\, \mathrm{d}\sigma(\theta);$$

see Aubin [3]. Equivalently,

$$m(\tilde{g}) = \lim_{\rho\to\infty} 4(n-1) \overline{\int_{S_{n-1}(1)}} H(1/\rho,\theta)\, \mathrm{d}\sigma(\theta).$$

If $2\omega \leqslant n - 6$, we saw (Aubin [4]) that this integral goes to $+\infty$. In all the other cases, $m(\tilde{g})$ is finite and positive, except for the sphere.

Theorem 2. *In this expression, the mass makes sense only if* $n < 6$ *or if* $2\omega > n - 6$. *Then* $\tilde{g}$ *is asymptotically Euclidean* (*Definition* 2) *of order* $2 + \omega$ *when* $2\omega > n - 6$ *if* $n \geqslant 4$, *and of order* $n - 2 = 1$ *if* $n = 3$.

3. The Mass in the sense of the Physicists: ADM mass [1,2]

It is generally agreed that the mathematical problem solved in [5] and the problem of the Physicists concerning the mass are the same. However, actually, it is not so simple, and we will give below, in all generality, the proof of this fact. In fact, in so doing, we prove an *inequality that has been long awaited by Physicists*: $m \geqslant 4(n-1)A$ with a constant $A > 0$. For an admissible conformal metric $\hat{g}$ to $\tilde{g}$, $\hat{m} \geqslant 4(n-1)A$.

Conversely, let us consider a complete manifold $(W, \tilde{g})$ such that, K being a compact set, $W\backslash K$ is diffeomorphic (by ξ) to one or many copies of the pointed ball of R^n: $B = B_0(\delta)\backslash\{0\}$ identified also to $B_P(\delta)\backslash\{P\}$. Let Ω be one of them. After this study it will be clear that the argument is the same for one end or several ends.

Definition 2. The manifold is *asymptotically Euclidean* in (Ω, ξ) if:

$$\tilde{g} - \mathcal{E} = \mathrm{O}\left(\rho^{-\tau}\right), \quad \partial_i \tilde{g}_{jk} = \mathrm{O}\left(\rho^{-\tau-1}\right), \quad \partial_{ij} \tilde{g}_{kl} = \mathrm{O}\left(\rho^{-\tau-2}\right), \quad \text{with } \tau > 0.$$

Remark 2. According to Bartnik [7], the mass makes sense if and only if $\tau > (n-2)/2$. This agrees with our result: the mass is a finite real number if and only if $\tau = 2 + \omega > 2 + (n-6)/2 = (n-2)/2$.

The usual hypotheses made by the Physicists for defining mass are:

(a) $\tilde{g} = (1 + \tilde{A}\rho^{2-n})^{4/(n-2)}\, (\mathcal{E} + \tilde{h})$ with $\tilde{h} = \mathrm{O}(\rho^{1-n})$
(b) $\tilde{R}$ integrable and non-negative.

This implies $\tilde{\psi}(\rho) = \overline{\int}_{\partial B(\rho)} \tilde{R}\, \mathrm{d}\tilde{\sigma} = \mathrm{O}(\rho^{-n-1})$, for otherwise $\tilde{R}$, which is non-negative, would not be integrable.

Expressing $\tilde{R}$ in terms of the derivatives of $\tilde{g}$ on Ω, we find:

$$\int_{B(\rho)\backslash B(\rho_o)} \tilde{R}\, \mathrm{d}\tilde{V} = \gamma(\rho) - \gamma(\rho_o) + \mathrm{O}\left(\rho^\alpha\right),$$

with $\alpha = n - 2(\tau + 1) < 0$. It follows that $m(\tilde{g}) = \lim_{\rho\to\infty} \gamma(\rho)$ is finite when $\tilde{R}$ is integrable. The hypothesis $\tilde{h} = \mathrm{O}(\rho^{1-n})$ implies the integrability of $\tilde{R}$.

Let us consider the conformal metric $g = \varphi^{4/(n-2)}\tilde{g}$, with $\varphi \in C^\infty$ such that $\Delta\varphi \geqslant 0$ on V and $\Delta\varphi > 0$ on $B(2/\delta)\backslash B(1/\delta)$, $\varphi = k = Cte$ for $\rho < 1/\delta$, and with principal part $\varphi = 1/(\rho^{n-2} + \tilde{A})$ for $\rho > 2/\delta$. By means of the inverse mapping $\rho \to r = 1/\rho$, the metric g on B extends itself on V with $\varphi = k$ outside B. It is of the type studied in [5]. Indeed $R \geqslant 0$ and $R > 0$ on $B(\delta)\backslash B(\delta/2)$ since $\tilde{R} \geqslant 0$, implies that L is invertible.

A point of coordinates $(\rho,\theta) \in \Omega$, $\rho > 1/\delta$, $\theta \in S_{n-1}(1)$, goes through ξ to a point (r,θ) in B ($r = 1/\rho < \delta$). In the system (r,θ), $g = (\xi^{-1})^*[\varphi^{4/(n-2)}\tilde{g}] = \mathcal{E} + h$, with $h = \mathrm{O}(r^{n-1})$.

This implies $\omega \geqslant n-3$. We have easily $2\omega > n-6$. It follows that $G = r^{2-n} + A + \mathrm{O}(r)$ with $A > 0$.

Let us now consider a change of conformal metric as above. Then we get on Ω a metric $\hat{g} = (1 + A\rho^{2-n})^{4/(n-2)} \times (\mathcal{E} + \hat{h})$, the Green function being $\hat{G} = 1/(\rho^{n-2} + A)$ and $\hat{R} \equiv 0$. Since by hypothesis $\tilde{R} \geqslant 0$, we must have $\tilde{A} \geqslant A$. Consequently, $m = 4(n-1)\tilde{A} \geqslant 4(n-1)A$.

Theorem 3. *Let $(W_n, \tilde{g})$ be asymptotically Euclidean ($n \geqslant 3$) (not conformal Euclidean) with scalar curvature $\tilde{R} \geqslant 0$ integrable. On each end (Ω_i, ξ), we suppose that the metric is of the form $\tilde{g}(\rho,\theta) = (1 + \tilde{A}_i\rho^{2-n})(\mathcal{E} + h_i)$ with $h_i = \mathrm{O}(\rho^{1-n})$. Then each mass m_i satisfies $m_i \geqslant 4(n-1)A_i > 0$, except if $\tilde{g} = \mathcal{E}$.*

4. The general case

We saw that, if the dimension $n \geqslant 6$, $H(P,Q)$ has no reason to be bounded, it may have singularities at P.

Let $\mathcal{M}$ denote the set of all metrics $\tilde{g}$ on W that are asymptotically Euclidean of order $> (n-2)/2$ with invertible conformal Laplacian. We suppose that there is a metric $\tilde{g} \in \mathcal{M}$ conformal to the initial metric such that $\tilde{R} \geqslant 0$ outside Ω and with mean value $\tilde{\psi}(\rho)$ integrable on Ω (above we saw that this is necessary in order the mass be finite), and that

$$(*)\quad \tilde{g} = (1 + \tilde{A}\rho^{2-n})^{4/(n-2)}(\mathcal{E} + \tilde{h}) \text{ with } \tilde{h} = \mathrm{O}(\rho^{-\tau}),\ \tau > (n-2)/2 \text{ and with an } \mathrm{O}(\rho^{1-n}) \text{ mean integral on } S_{n-1}(1).$$

We do the same at each end if there are several ends.

Then we proceed as above. With a change of conformal metric on W, we obtain on V (by extension) a metric g to which there correspond at P (resp. P_i), A (resp. A_i) positive real numbers (except on S_n for which $A = 0$). Coming back to W, we obtain a metric in $\mathcal{M}$, namely:

$$\hat{g} = G^{4/(n-2)}g = \left(1 + A\rho^{2-n}\right)^{4/(n-2)}(\mathcal{E} + \hat{h}) \quad \text{with } \hat{R} = 0 \text{ and } A > 0.$$

Moreover, $\tilde{\psi} \geqslant 0$, thus $m > 0$. Indeed $\tilde{R}$ can be expressed itself in terms of $\hat{R}$. Integrating this expression, we get that $\tilde{\psi} - \psi$ is of the same sign as $\tilde{A} - A$ (like the end of 3.). Therefore:

$$m = 4(n-1)\tilde{A} \geqslant 4(n-1)A > 0.$$

Theorem 4. *Let $(W, \tilde{g}) \in \mathcal{M}$ be of the type $(*)$ ($\tilde{g}$ not conformally Euclidean). Then its mass m is positive. Moreover, $m \geqslant 4(n-1)A$, where the real number $A > 0$ corresponds to $\tilde{h}$ (resp. $A_i > 0$ at each end).*

Remark. This Note is more than an announcement: it contains all the proofs.

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