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C. R. Acad. Sci. Paris, Ser. I 345 (2007) 695–698
COMPTES RENDUS MATHEMATIQUE
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Dynamical Systems

The moduli space of germs of generic families of analytic diffeomorphisms unfolding a parabolic fixed point ☆

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Received 27 June 2007; accepted after revision 13 October 2007

Available online 26 November 2007

Presented by Étienne Ghys

Abstract

We describe the moduli space of germs of generic families of analytic diffeomorphisms which unfold a parabolic fixed point of codimension 1. A complete modulus is given by unfolding the Écalle–Voronin modulus over a sector of opening greater than 2π in the canonical parameter ϵ. In the region of overlap (Glutsyuk sector of parameter space) where the two fixed points are connected by orbits, we identify the necessary compatibility between the two representatives of the modulus. The compatibility condition implies the existence of a normalization for which the modulus is $\frac{1}{2}$-summable in ϵ, non-summability occurring in the direction of real multipliers of the fixed points. We show that the compatibility condition together with the summability is sufficient for realization of the modulus. ***To cite this article: C. Christopher, C. Rousseau, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

L'espace des modules des germes de familles génériques de difféomorphismes analytiques déployant un point fixe parabolique. On donne l'espace des modules des germes de familles génériques de difféomorphismes analytiques déployant un point fixe parabolique de codimension 1. Un module complet est donné par le déploiement du module d'Écalle–Voronin sur un secteur d'ouverture plus grande que 2π du paramètre canonique. Dans le sous-secteur recouvert deux fois (sous-secteur Glutsyuk), là où les deux points fixes sont connectés par des orbites, on identifie une condition de compatibilité nécessaire satisfaite par les deux représentants du module. Cette condition implique l'existence d'une normalisation sous laquelle le module est $\frac{1}{2}$-sommable en ϵ, la non-sommabilité se produisant dans la direction des multiplicateurs réels aux points fixes. On montre que la condition de compatibilité, jointe à cette propriété de sommabilité, est suffisante pour réaliser le module. ***Pour citer cet article : C. Christopher, C. Rousseau, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

☆ This work is supported by NSERC in Canada.
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doi:10.1016/j.crma.2007.10.033

1. Statement of the results

We consider germs of generic analytic 1-parameter families of diffeomorphisms of $(\mathbb{C},0)$ unfolding a parabolic point. Such families can be 'prepared' so that the parameter becomes an analytic invariant [2]. A prepared family with canonical (complex) parameter ϵ has the form

$$f_\epsilon(z) = z + (z^2-\epsilon)\big(1+b(\epsilon)+c(\epsilon)z+\mathrm{O}(z^2-\epsilon)\big), \tag{1}$$

so that $1/\sqrt{\epsilon} = 1/\ln(f'_\epsilon(\sqrt{\epsilon})) - 1/\ln(f'_\epsilon(-\sqrt{\epsilon}))$. The formal invariant, $a(\epsilon)$, is the analytic function in ϵ given by $a(\epsilon) = 1/\ln(f'_\epsilon(\sqrt{\epsilon})) + 1/\ln(f'_\epsilon(-\sqrt{\epsilon}))$.

In [2] (and [5]) it is shown that a modulus for such unfoldings can be obtained from the formal parameter, $a(\epsilon)$, and two analytic functions, $\Psi^0_{\hat\epsilon}$ and $\Psi^\infty_{\hat\epsilon}$, which describe the shift between two choices of Fatou coordinates on their region of overlap. The Fatou coordinates are chosen so that

$$\lim_{\operatorname{Im} W\to-\infty} \Psi^0_{\hat\epsilon} = \mathrm{id}, \qquad \lim_{\operatorname{Im} W\to+\infty} \Psi^\infty_{\hat\epsilon} = T_{-2\pi i a(\epsilon)}, \tag{2}$$

where T_b denotes translation by b. Here, $\hat\epsilon$ represents the parameter ϵ when lifted to the universal cover of the punctured disc. These functions are always assumed to satisfy $T_1 \circ \Psi^{0,\infty}_{\hat\epsilon} = \Psi^{0,\infty}_{\hat\epsilon} \circ T_1$. That is, they could also be considered as lifts of maps $\psi^{0,\infty}_{\hat\epsilon}$ between neighborhoods of the poles of two spheres. We demonstrate the following theorems, which identify the extra conditions on the $\Psi^{0,\infty}_{\hat\epsilon}$ to form a moduli space for analytic unfoldings, and prove their sufficiency:

Theorem 1.1. *Consider a prepared germ of the form* (1). *Then for $\delta \in (0,\pi)$ there exists $\rho > 0$ and a representative of the modulus, $(\Psi^0_{\hat\epsilon}, \Psi^\infty_{\hat\epsilon})$, defined for $\hat\epsilon \in V_{\delta,\rho} = \{\hat\epsilon;\, |\hat\epsilon| < \rho, \arg\hat\epsilon \in (-\delta, 2\pi+\delta)\}$ such that.*

(i) *there exists $Y_0 > 0$ such that $\Psi^0_{\hat\epsilon}$ (resp. $\Psi^\infty_{\hat\epsilon}$) is analytic on* $\operatorname{Im} W < -Y_0$ (*resp.* $\operatorname{Im} W > Y_0$) *for all $\hat\epsilon \in V_{\delta,\rho}$;*
(ii) *$\Psi^{0,\infty}_{\hat\epsilon}$ are $\frac{1}{2}$-summable in ϵ (i.e.* 1*-summable in $\sqrt{\hat\epsilon}$) with direction of non-summability given by $\mathbb{R}^+$ (see* [3] *for k-summability*);
(iii) *$\Psi^{0,\infty}_{\hat\epsilon}$ give rise to the following compatibility condition.*

We cover the region of overlap $\arg(\epsilon) \in (-\delta,\delta)$ (*Glutsyuk sector*) *by the two subsectors*

$$\overline{V} = \big\{\hat\epsilon;\, 0<|\hat\epsilon|<\rho, \arg\hat\epsilon \in (-\delta,+\delta)\big\}, \qquad \widetilde{V} = \big\{\hat\epsilon;\, 0<|\hat\epsilon|<\rho, \arg\hat\epsilon \in (2\pi-\delta, 2\pi+\delta)\big\}, \tag{3}$$

on which we denote $\hat\epsilon$ by $\bar\epsilon$ and $\tilde\epsilon$ respectively (and $\Psi^{0,\infty}_\epsilon$ by $\widetilde{\Psi}^{0,\infty}_{\tilde\epsilon}$ and $\overline{\Psi}^{0,\infty}_{\bar\epsilon}$). We also define

$$\alpha^0 = -2\pi i\big(1-a(\epsilon)\sqrt{\hat\epsilon}\big)/2\sqrt{\hat\epsilon}, \qquad \alpha^\infty = -2\pi i\big(1+a(\epsilon)\sqrt{\hat\epsilon}\big)/2\sqrt{\hat\epsilon}.$$

On $\widetilde{V}$ (resp. $\overline{V}$), $\alpha^{0,\infty}$ takes values $\tilde\alpha^{0,\infty}$ (resp. $\bar\alpha^{0,\infty}$). Then there exist maps $\overline{H}^{0,\infty}_{\bar\epsilon}$ and $\widetilde{H}^{0,\infty}_{\tilde\epsilon}$ holomorphic in W and $\hat\epsilon$ and commuting with T_1 which satisfy

$$\text{on } \widetilde{V} \quad \begin{cases} \widetilde{H}^0_{\tilde\epsilon} \circ T_{\tilde\alpha^0} \circ \widetilde{\Psi}^0_{\tilde\epsilon} = T_{\tilde\alpha^0} \circ \widetilde{H}^0_{\tilde\epsilon}, \\ \widetilde{H}^\infty_{\tilde\epsilon} \circ T_{\tilde\alpha^0} \circ \widetilde{\Psi}^\infty_{\tilde\epsilon} = T_{\tilde\alpha^\infty} \circ \widetilde{H}^\infty_{\tilde\epsilon}, \end{cases} \qquad \text{on } \overline{V} \quad \begin{cases} \overline{H}^0_{\bar\epsilon} \circ \overline{\Psi}^0_{\bar\epsilon} \circ T_{\bar\alpha^0} = T_{\bar\alpha^0} \circ \overline{H}^0_{\bar\epsilon}, \\ \overline{H}^\infty_{\bar\epsilon} \circ \overline{\Psi}^\infty_{\bar\epsilon} \circ T_{\bar\alpha^0} = T_{\bar\alpha^\infty} \circ \overline{H}^\infty_{\bar\epsilon}. \end{cases} \tag{4}$$

The compatibility condition is

$$\widetilde{H}^\infty_{\tilde\epsilon} \circ \big(\widetilde{H}^0_{\tilde\epsilon}\big)^{-1} = T_{2\pi i a(\epsilon)} \circ \overline{H}^0_{\bar\epsilon} \circ \big(\overline{H}^\infty_{\bar\epsilon}\big)^{-1} \circ T_{D(\epsilon)} \tag{5}$$

for some constant $D(\epsilon) = -2\pi i a(\epsilon) + \mathrm{O}(\exp(-\frac{A}{|\sqrt{\epsilon}|}))$ with $A > 0$.

Conversely, we have the following:

Theorem 1.2. *Consider a germ of a function $a(\epsilon)$ analytic in ϵ and a germ of a family $(\Psi^0_{\hat\epsilon}, \Psi^\infty_{\hat\epsilon})_{\hat\epsilon\in V_{\delta,\rho}}$ for some $\delta \in (0,\pi)$ and $\rho > 0$, which satisfies* (i), (ii) *and* (iii) *of Theorem* 1.1. *Then there exists a germ of family of analytic diffeomorphisms*

$$f_\epsilon = z + (z^2-\epsilon)\big(1+\mathrm{O}(\epsilon)+\mathrm{O}(z)\big) \tag{6}$$

whose modulus over a fixed neighborhood in (z,ϵ)*-space is given by* $(\Psi^0_{\hat\epsilon}, \Psi^\infty_{\hat\epsilon})_{\hat\epsilon\in V_{\delta,\rho}}$ *and* $a(\epsilon)$*.*

Remark 1. In the unfolded modulus, the map $\Psi^0_{\hat\epsilon}$ (resp. $\Psi^\infty_{\hat\epsilon}$) refers to the dynamics near $-\sqrt{\hat\epsilon}$ (resp. $\sqrt{\hat\epsilon}$). When $\hat\epsilon$ makes a full turn, $-\sqrt{\hat\epsilon}$ and $\sqrt{\hat\epsilon}$ are exchanged. We therefore have two different ways of describing the dynamics near each singular point in the region of overlap. The functions $\widetilde{H}^{0,\infty}_{\hat\epsilon}$ and $\overline{H}^{0,\infty}_{\hat\epsilon}$ are lifts of normalizations of $f_{\hat\epsilon}$ at the fixed points. Such maps must exist due to the hyperbolicity of the fixed points in the region of overlap [1]. The compatibility condition guarantees that $\widetilde{H}^{0,\infty}_{\hat\epsilon}$ in $\widetilde{V}$ and $\overline{H}^{0,\infty}_{\hat\epsilon}$ in $\overline{V}$ glue to the same dynamics. It is an interesting fact that though the functions $\widetilde{H}^{0,\infty}_{\hat\epsilon}$ and $\overline{H}^{0,\infty}_{\hat\epsilon}$ have no geometrical meaning at $\epsilon=0$, yet their limits can be calculated and are well-behaved. This fact is of some importance in step (2) of the proof below.

Remark 2. The correspondence established in Theorems 1.1 and 1.2 can be extended in a natural way to the case when f_ϵ depends on a number of auxiliary analytic parameters.

2. The proofs

The proof of Theorems 1.1 and 1.2 is composed of four parts:

(1) the construction of $(\Psi^{0,\infty}_{\hat\epsilon})_{\hat\epsilon\in V_{\delta,\rho}}$: this is done in [2];
(2) the derivation of the compatibility condition from which the $\frac{1}{2}$-summability of $\Psi^{0,\infty}_{\hat\epsilon}$ follows;
(3) the 'local realization' over sectorial neighborhoods in $\hat\epsilon$ of small opening: this yields the realization by a family $g_{\hat\epsilon}$ ramified in ϵ with uniform limit g_0 when $\hat\epsilon\to 0$ along any ray $\arg\hat\epsilon = Const$. The local realization does not use (2);
(4) the global realization: from $g_{\hat\epsilon}$, using the compatibility condition we construct a uniform family f_ϵ over an abstract 2-dimensional manifold. The $\frac{1}{2}$-summability of $\Psi^{0,\infty}_{\hat\epsilon}$ allows application of the Newlander–Nirenberg theorem to show that this manifold is an open set of $\mathbb{C}^2$.

In more detail:
(2) To decide when two diffeomorphisms f_ϵ unfolding a parabolic point are conjugate, we embed them in the flow of the vector field $v_\epsilon = \frac{z^2-\epsilon}{1+a(\epsilon)z}\frac{d}{dz}$ on adequate sectorial domains and measure the obstruction to a global embedding. For this it is easier to work in a coordinate W which is the time of the vector field v_ϵ. The four maps $\widetilde{H}^{0,\infty}_{\hat\epsilon}$ and $\overline{H}^{0,\infty}_{\hat\epsilon}$ can be seen as the lifting of a change of coordinate embedding the map in a flow near the hyperbolic fixed points $\pm\sqrt{\epsilon}$. In the W-coordinate this flow is that of $\frac{\partial}{\partial W}$. The comparison of these embeddings is an invariant: this is exactly what is expressed by the compatibility condition, modulo a scaling given by the constant $D(\epsilon)$. To derive the $\frac{1}{2}$-summability of $\Psi^0_{\hat\epsilon}$ we explicitly calculate the maps $\widetilde{H}^{0,\infty}_{\hat\epsilon}$ and $\overline{H}^{0,\infty}_{\hat\epsilon}$ in some region inside $\operatorname{Im} W > Y_0$. There exists $A, C > 0$ such that

$$\begin{cases} |\overline{H}^0_{\hat\epsilon} - \mathrm{id}| < C\exp(-A/|\sqrt{\hat\epsilon}|), \\ |\widetilde{H}^0_{\hat\epsilon} - \mathrm{id}| < C\exp(-A/|\sqrt{\hat\epsilon}|), \end{cases} \quad \text{and} \quad \begin{cases} |(\overline{H}^\infty_{\hat\epsilon})^{-1} - \overline{\Psi}^\infty_{\hat\epsilon} - 2\pi i a| < C\exp(-A/|\sqrt{\hat\epsilon}|), \\ |\widetilde{H}^\infty_{\hat\epsilon} - \widetilde{\Psi}^\infty_{\hat\epsilon} - 2\pi i a| < C\exp(-A/|\sqrt{\hat\epsilon}|), \end{cases}$$

so the compatibility condition yields the $\frac{1}{2}$-summability of $\Psi^\infty_{\hat\epsilon}$ by the theorem of Ramis–Sibuya [4]. A similar study in some region inside $\operatorname{Im} W < -Y_0$ allows to prove the $\frac{1}{2}$-summability of $\Psi^0_{\hat\epsilon}$.
(3) The local realization for $\hat\epsilon$ in a sector of small opening is first done for fixed $\hat\epsilon$ by gluing two sectors $U^\pm_{\hat\epsilon}$ as in Fig. 1 with maps $\Xi^{0,\infty}_{\hat\epsilon}$ constructed from $\Psi^{0,\infty}_{\hat\epsilon}$ ($\Xi^{0,\infty}_{\hat\epsilon}$ is the conjugate of $\Psi^{0,\infty}_{\hat\epsilon}$ under the change $z\mapsto W$). The $U^\pm_{\hat\epsilon}$ are images of modified strips in the W domain. This yields a complex manifold $M_{\hat\epsilon}$ which we endow with a diffeomorphism given on each $U^\pm_{\hat\epsilon}$ by the time-one map of the vector field v_ϵ. The theorem of Ahlfors–Bers allows recognition of the abstract manifold $M_{\hat\epsilon}$ as a neighborhood of $\pm\sqrt{\epsilon}$ in $\mathbb{C}$ and to fill the holes at the fixed points. The construction can be made to depend analytically on $\hat\epsilon$ with a continuous fixed limit M_0 as $\hat\epsilon\to 0$ along a ray. In this way we realize the modulus in a ramified family $g_{\hat\epsilon}$ for $\hat\epsilon$ in some $V_{\delta,\rho}$.

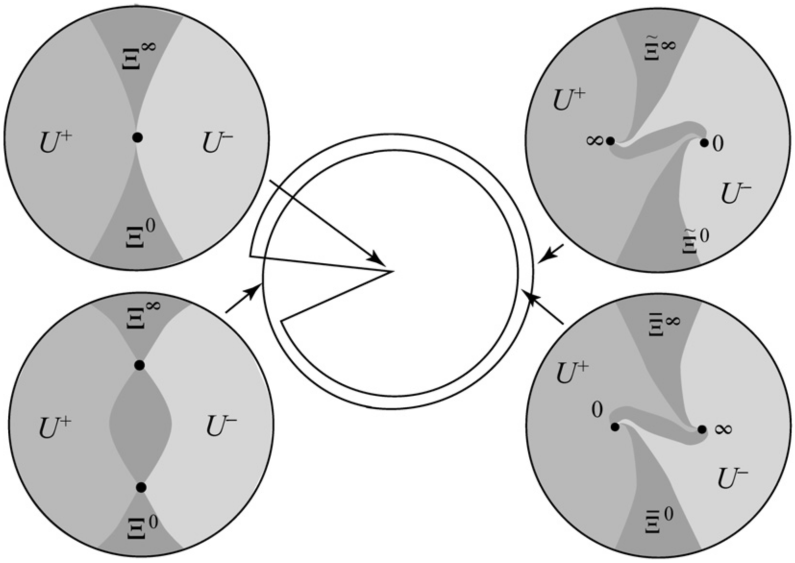

Fig. 1. The sectors $U^{\pm}_{\hat{\epsilon}}$ for $\hat{\epsilon} \in V_{\delta,\rho}$: $U^{-}_{\hat{\epsilon}}$ is light grey, $U^{+}_{\hat{\epsilon}}$ is middle grey and their intersection is dark grey.

(4) The compatibility condition ensures that $g_{\bar{\epsilon}}$ and $g_{\tilde{\epsilon}}$ are conjugate by some $J_{\hat{\epsilon}}$ for $\arg \bar{\epsilon} \in (-\delta, \delta)$ and $\tilde{\epsilon} = e^{2\pi i}\bar{\epsilon}$. Moreover $g_{\hat{\epsilon}}$ is $\frac{1}{2}$-summable in $\hat{\epsilon}$ with directions of non-summability given by $\arg \hat{\epsilon} \in \{0, 2\pi\}$. An indirect consequence of this is that $|J_{\hat{\epsilon}}| = \mathrm{O}(\exp(-\frac{A}{|\sqrt{\hat{\epsilon}}|}))$ for some positive A. We construct an abstract manifold by gluing $B(0,r) \times \{\arg \hat{\epsilon} \in (-\delta, \delta)\}$ with $B(0,r) \times \{\arg \hat{\epsilon} \in (2\pi - \delta, 2\pi + \delta)\}$, by means of $(z, \bar{\epsilon}) \mapsto (J_{\hat{\epsilon}}(z), \tilde{\epsilon})$ and pasting in $B(0,r) \times \{\epsilon = 0\}$ to fill the hole. This gives us a C^{∞} manifold $\mathcal{M}$ with an almost complex structure. We apply the Newlander–Nirenberg theorem to realize $\mathcal{M}$ as a neighborhood of the origin in $\mathbb{C}^2$ on which $g_{\hat{\epsilon}}$ induces a family of analytic diffeomorphisms f_{ϵ} realizing the modulus.

Acknowledgements

We are grateful to Reinhard Schäfke, Loïc Teyssier and the referee for helpful discussions or remarks.

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