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Group Theory

On the number of conjugates defining the solvable radical of a finite group

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Abstract

We are looking for the smallest integer $k > 1$ providing the following characterization of the solvable radical $R(G)$ of any finite group G: $R(G)$ consists of the elements g such that for any k elements $a_1, a_2, \ldots, a_k \in G$ the subgroup generated by the elements $g, a_i g a_i^{-1}$, $i = 1, \ldots, k$, is solvable. Our method is based on considering a similar problem for commutators: find the smallest integer $\ell > 1$ with the property that $R(G)$ consists of the elements g such that for any ℓ elements $b_1, b_2, \ldots, b_\ell \in G$ the subgroup generated by the commutators $[g, b_i]$, $i = 1, \ldots, \ell$, is solvable. ***To cite this article: N. Gordeev et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Sur le nombre de conjugués définissant le radical résoluble d'un groupe fini. Nous cherchons le plus petit entier $k > 1$ caractérisant le radical résoluble $R(G)$ d'un groupe fini G comme suit : $R(G)$ est l'ensemble des éléments g tels que pour toute partie à k éléments $\{a_1, a_2, \ldots, a_k\} \subset G$ le sous-groupe engendré par les élements $g, a_i g a_i^{-1}$, $i = 1, \ldots, k$, est résoluble. Notre méthode s'appuie sur la considération d'un problème similaire pour les commutateurs. Nous cherchons le plus petit entier $\ell > 1$ ayant la propriété suivante : $R(G)$ est l'ensemble des éléments g tels que pour toute partie à ℓ éléments $\{b_1, b_2, \ldots, b_\ell\} \subset G$ le sous-groupe engendré par les commutateurs $[g, b_i]$, $i = 1, \ldots, \ell$, est résoluble. ***Pour citer cet article : N. Gordeev et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

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Notre but est une nouvelle caractérisation du radical résoluble $R(G)$ d'un groupe fini G.

Théorème 0.1. *Le radical résoluble d'un groupe fini G est l'ensemble des éléments g satisfaisant la propriété suivante : pour toute partie à 7 éléments $\{a_1, a_2, \ldots, a_7\} \subset G$, le sous-groupe engendré par les éléments $g, a_i g a_i^{-1}$, $i = 1, \ldots, 7$, est résoluble.*

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La démonstration utilise la classification des groupes simples finis.
Cet énoncé implique la caractérisation suivante des groupes résolubles finis :

Théorème 0.2. *Un groupe fini G est résoluble si et seulement si dans chaque classe de conjugaison de G toute partie à* 8 *éléments engendre un sous-groupe résoluble.*

Nous espérons d'améliorer ces caractérisations.

Conjecture 0.3. *Le radical résoluble d'un groupe fini G est l'ensemble des éléments g satisfaisant la propriété suivante* : *pour toute partie à trois éléments $\{a, b, c\} \subset G$, le sous-groupe engendré par les éléments g, aga^{-1}, bgb^{-1}, cgc^{-1} est résoluble.*

Cette assertion implique :

Conjecture 0.4. *Un groupe fini G est résoluble si et seulement si dans chaque classe de conjugaison de G toute partie à quatre éléments engendre un sous-groupe résoluble.*

Remarque 0.5. Ces caractérisations sont les meilleures possibles : dans les groupes symétriques S_n ($n \geqslant 5$) trois transpositions engendrent un sous-groupe résoluble.

Remarque 0.6. Le point clé dans notre démonstration du Théorème 0.1 est le Théorème 0.8 ci-dessous. Pour démontrer la Conjecture 0.3 (et donc la Conjecture 0.4), il faut étendre l'énoncé du Théorème 0.8 à tous les groupes *presque simples* (c'est-à-dire, aux groupes H tels que $G \subseteq H \subseteq \mathrm{Aut}(G)$ pour certain groupe simple G).

Définition 0.7. Soit $k \geqslant 2$ un nombre entier. On dit que $g \in G$ est un élément *k-radiciel* si pour toute partie à k éléments $\{a_1, \ldots, a_k\} \subset G$, le sous-groupe engendré par les commutateurs $[a_1, g], \ldots, [a_k, g]$ est résoluble.

Nous démontrons :

Théorème 0.8. *Soit G un groupe simple fini non abélien. Alors G ne contient pas d'élément* 3*-radiciel non trivial.*

Ce théorème implique les Théorèmes 0.1 and 0.2.
La preuve consiste à faire une analyse cas par cas (groupes alternés, groupes de type de Lie, groupes sporadiques) ; dans certains cas on obtient un énoncé plus précis.

Définition 0.9. Notons $\rho(g)$ le plus petit entier n ayant la propriété suivante : $g \notin R(G)$ si et seulement s'il existe $x_1, \ldots, x_n \in G$ tels que le sous-groupe engendré par les commutateurs $[g, x_1], \ldots, [g, x_n]$ n'est pas résoluble. On appelle le nombre $\rho(G) := \max_{g \in G \setminus R(G)} \rho(g)$ le degré radiciel de G.

Théorème 0.10. *Si G est un groupe simple fini non abélien, alors $\rho(G) \leqslant 3$. Si G est un groupe de type de Lie sur un corps K tel que* $\mathrm{char}\, K \neq 2$ *et $K \neq \mathbb{F}_3$, ou un groupe sporadique non isomorphe à Fi_{22} ou Fi_{23}, alors $\rho(G) = 2$.*

1. Main results

Our goal is to prove a new characterization of the solvable radical $R(G)$ of a finite group G.

Theorem 1.1. *The solvable radical of any finite group G consists of the elements g satisfying the property*: *for any* 7 *elements $a_1, a_2, \ldots, a_7 \in G$ the subgroup generated by the elements $g, a_i g a_i^{-1}$, $i = 1, \ldots, 7$, is solvable.*

The proof involves the classification of finite simple groups.
This theorem implies the following characterization of finite solvable groups:

Theorem 1.2. *A finite group G is solvable if and only if in each conjugacy class of G every* 8 *elements generate a solvable subgroup.*

We hope to strengthen these results.

Conjecture 1.3. *The solvable radical of a finite group G consists of the elements g satisfying the property: for any* 3 *elements $a, b, c \in G$ the subgroup generated by conjugates $g, aga^{-1}, bgb^{-1}, cgc^{-1}$ is solvable.*

This statement implies:

Conjecture 1.4. *A finite group G is solvable if and only if in each nontrivial conjugacy class of G every four elements generate a solvable subgroup.*

Remark 1.5. These characterizations are the best possible: in the symmetric groups S_n $(n \geqslant 5)$ any triple of transpositions generates a solvable subgroup.

Remark 1.6. The main step in our proof of Theorem 1.1 is Theorem 1.8. To prove Conjecture 1.3 (and hence Conjecture 1.4), one has to extend the statement of Theorem 1.8 to all *almost simple* groups, i.e. to the groups H such that $G \subseteq H \subseteq \mathrm{Aut}(G)$ for some simple group G (see Remark 2.1 below).

Throughout the note $\langle a_1, \ldots, a_k \rangle$ stands for the subgroup of G generated by $a_1, \ldots, a_k \in G$. We define the commutator of $x, y \in G$ by $[x, y] = xyx^{-1}y^{-1}$.

Definition 1.7. Let $k \geqslant 2$ be an integer. We say that $g \in G$ is a *k-radical element* if for any $a_1, \ldots, a_k \in G$ the subgroup $H = \langle [a_1, g], \ldots, [a_k, g] \rangle$ is solvable.

We prove

Theorem 1.8. *Let G be a finite non-Abelian simple group. Then G does not contain nontrivial* 3*-radical elements.*

This theorem implies Theorems 1.1 and 1.2.

The proof goes by case-by-case inspection (alternating groups, groups of Lie type, sporadic groups); in some cases the statement is more precise.

Definition 1.9. Denote by $\rho(g)$ the smallest possible n with the following property: $g \notin R(G)$ if and only if there exist $x_1, \ldots, x_n \in G$ such that the subgroup $\langle [g, x_1], \ldots, [g, x_n] \rangle$ is not solvable. We call the number $\rho(G) := \max_{g \in G \setminus R(G)} \rho(g)$ the radical degree of G.

Theorem 1.10. *If G is a finite non-Abelian simple group, then $\rho(G) \leqslant 3$. If G is a group of Lie type over a field K with* $\mathrm{char}\, K \neq 2$ *and $K \neq \mathbb{F}_3$, or a sporadic group not isomorphic to Fi_{22}, Fi_{23}, then $\rho(G) = 2$.*

The characterizations of the solvable radical described above should be compared with earlier results. Namely, let $F_2 = F(x, y)$ denote the free two generator group. Define a sequence $\vec{e} = e_1, e_2, e_3, \ldots,$ where $e_i(x, y) \in F_2$, by

$$e_1(x, y) = [x, y], \ldots, e_n(x, y) = \big[e_{n-1}(x, y), y\big], \ldots.$$

An element $g \in G$ is called an Engel element if for every $a \in G$ there exists $n = n(a, g)$ such that $e_n(a, g) = 1$.

In 1957 R. Baer proved the following theorem [2,13]:

Theorem 1.11. *The nilpotent radical of a Noetherian group G coincides with the collection of all Engel elements of G.*

In particular, Baer's theorem holds for finite groups. Similar theorems have been established for many classes of infinite groups satisfying some additional conditions (see, for example, [18,17]).

A tempting but difficult problem is to find a counterpart of Baer's theorem for the solvable radical of a finite group. In other words, one has to find an Engel-like sequence $\vec{u} = \vec{u}_n(x, y)$ such that $g \in G$ belongs to the solvable radical $R(G)$ if and only if for every $a \in G$ there exists $n = n(a, g)$ such that $u_n(a, g) = 1$. First results towards a solution of this problem have been obtained in [4–6,3].

In [12] a Thompson-like characterization of the solvable radical of a finite group has been obtained (it also holds for any linear group or a PI-group):

Theorem 1.12. *[12] The solvable radical $R(G)$ of a finite group G coincides with the set of all elements $g \in G$ such that for any $a \in G$ the subgroup generated by g and a is solvable.*

This theorem can be viewed as an implicit description of the solvable radical since it does not provide any explicit formulas which determine if a particular element belongs to $R(G)$. The results of the present paper can be viewed as a next step to a more explicit characterization of the solvable radical.

2. Theorem 1.8 implies Theorem 1.1

Suppose Theorem 1.8 holds. Let us prove Theorem 1.1.

For brevity, let us call the elements $g \in G$ satisfying the condition of Theorem 1.1, *suitable*. Clearly, all the elements of the radical $R(G)$ are suitable. We want to show that any suitable element g belongs to $R(G)$. We may assume that G is semisimple (i.e. $R(G) = 1$).

Recall that any finite semisimple group G contains a unique maximal normal centreless completely reducible (CR) subgroup (by definition, CR means a direct product of finite non-Abelian simple groups) which is called the CR-radical of G (see [19, 3.3.16]). We call a product of the isomorphic factors in the decomposition of the CR-radical *an isotypic component* of G. Denote the CR-radical of G by $V(G)$. This is a characteristic subgroup of G.

Let G be a minimal counterexample to the statement of Theorem 1.1. Then it is easy to show that $V(G)$ has only one isotypic component. Any $g \in G$ acts as an automorphism $\bar{g}$ on $V(G) = H_1 \times \cdots \times H_n$, where all H_i, $1 \leqslant i \leqslant n$, are isomorphic simple non-Abelian groups.

Suppose that g is a suitable element. Using the 'one and a half' generation theorem for simple groups by Guralnick and Kantor [11, Corollary of Th. I on p. 745], one can show that $\bar{g}$ cannot act on $V(G)$ as a nontrivial element of the symmetric group S_n. So we can assume that g acts as an automorphism of a simple group H. Consider the extension G_1 of the group H with the automorphism $\bar{g}$. Since G_1 has no centre, one can find $x \in H$ such that $[x, \bar{g}] \neq 1$. By Theorem 1.8 there exist $y_1, y_2, y_3 \in H$ such that the subgroup $\langle [[x, g], y_1], [[x, g], y_2], [[x, g], y_3] \rangle$ is not solvable. Using the formula $y[x, g]y^{-1} = [x, g][[g, x], y]$, we obtain

$$\big\langle \big[[x, g], y_1\big], \big[[x, g], y_2\big], \big[[x, g], y_3\big] \big\rangle \leqslant \big\langle g, x^{-1}gx, y_i^{-1}gy_i, y_i^{-1}x^{-1}gxy_i \mid i = 1, 2, 3 \big\rangle.$$

As g is suitable, the latter subgroup must be solvable. Contradiction with the choice of y_i.

Remark 2.1. The above argument shows that Conjectures 1.3 and 1.4 follow from the analogue of Theorem 1.8 for the almost simple groups.

3. Scheme of the proof of Theorem 1.8

3.1. Alternating groups

Proposition 3.1. *Let $G = A_n$, $n \geqslant 5$. Then $\rho(G) = 2$.*

Proof. We proceed by induction. For $n = 5, 6$ the statement can be checked in a straightforward manner. To make the induction step, we consider separately the cases where y can (or cannot) be represented in the form $y = \sigma\tau$, $\sigma \in A_m, \sigma \neq 1, m < n$. □

3.2. *Groups of Lie type,* $\mathrm{char}(K) \neq 2$ *and* $|K| \neq 3$

Let first G be a group of rank 1.

Proposition 3.2. *Let G be one of the groups*

$$A_1(q)\ (q \neq 2,3), \quad {}^2A_2(q^2)\ (q \neq 2), \quad {}^2B_2(2^{2m+1})\ (m \geqslant 1), \quad {}^2G_2(3^{2m+1})\ (m \geqslant 0).$$

Then $\rho(G) = 2$.

The proof involves calculations based on different canonical decompositions in G. The uniform part of the proof relies on the following theorem of Gow (compare with [7]) regarding conjugacy classes of semisimple elements in Chevalley groups:

Theorem 3.3. *[10] Let G be a finite simple group of Lie type, and let $g \neq 1$ be a semisimple element in G. Let L be a conjugacy class of G consisting of regular semisimple elements. Then there exist a regular semisimple $x \in L$ and $z \in G$ such that $g = [x, z]$.*

Certain steps require, however, explicit matrix representations for the groups of rank 1 (see [14,15]). As usual, groups over small fields are considered separately.

Suppose now that the rank of G is greater than 1.

Theorem 3.4. *Let G be a Chevalley group of rank > 1 over a field K with* $\mathrm{char}\, K \neq 2$, $K \neq \mathbb{F}_3$. *Then $\rho(G) = 2$.*

We use the Levi decomposition of G together with arguments from [9] (based on the notions of generalized Bruhat cells and generalized Coxeter elements) in order to reduce to rank 1 case.

3.3. *Groups of Lie type,* $\mathrm{char}(K) = 2$ *or* $|K| = 3$

Proposition 3.5. *Let G be a non-solvable Chevalley group over a field K, where either* $\mathrm{char}\, K = 2$ *or* $K = \mathbb{F}_3$. *Then* $\rho(G) \leqslant 3$.

The proof goes by reduction to the case of groups of rank at most 3 and involves more technicalities. In particular, the case $G = {}^2F_4(q)$ requires separate consideration. Some groups of small ranks are treated by MAGMA.

Note that the estimate of Proposition 3.5 is sharp as follows from the case of groups generated by 3-transpositions (see [8,1] for definitions and notations).

Proposition 3.6. *Let G be one of the following groups* (*with the notation of* [1]): *a symmetric group S_n; a symplectic group* $\mathrm{Sp}(2n, 2)$ $(n \geqslant 2)$; *an orthogonal group* $\mathrm{O}^{\mu}(2n, 2)$ *for* $\mu \in \{-1, 1\}$ *and* $n \geqslant 2$; *a unitary group* $\mathrm{PSU}(n, 2)$ $(n \geqslant 4)$; *an orthogonal group* $\mathrm{O}^{\mu,\pi}(n, 3)$ *for* $\mu \in \{-1, 1\}$, $\pi \in \{-1, 1\}$, *and* $n \geqslant 4$; *one of Fischer's groups* Fi_{22}, Fi_{23}, Fi_{24}. *Then G contains a nontrivial 2-radical element.*

3.4. *Sporadic groups*

Proposition 3.7. *Let G be a sporadic simple group. Then $\rho(G) = 3$ for $G = Fi_{22}, Fi_{23}$ and $\rho(G) = 2$ for all the remaining groups.*

For all groups except the Monster M, our main tool was MAGMA. We relied on the ATLAS classification of conjugacy classes of maximal cyclic subgroups [21]. For larger sporadic groups we had to replace most standard MAGMA procedures by our own ones in order to avoid storing the whole group and large subgroups. In particular, to check whether a subgroup under consideration is not solvable, we used the Hall–Thompson criterion [20]: a group H is non-solvable if and only if it contains elements a, b, c of pairwise coprime orders such that $abc = 1$.

Our proof for M also relies on the classification of conjugacy classes. However, all other arguments are theoretic. We reduce the statement to checking the elements g of prime orders p each of those (except for $p = 41$) can be included in some proper simple subgroup of M [16]. The case $p = 41$ is treated separately using the fact that the normalizer $N_M(\langle g\rangle) = 41 \cdot 40$ is a maximal subgroup of M [21].

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