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Q-superlinear convergence of the GMRES algorithm for multi-materials with strong interface

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Abstract

We consider a model non-classical transmission problem corresponding to a multistructure composed of two bodies bonded by a thin strong layer. By using a domain decomposition, the problem is reduced to an equation defined on the interface of the form $(\mathcal{I}-\mathcal{G})g=F$. We prove that $\mathcal{G}$ is compact of Carleman class C_s, and hence the q-superlinearly convergence of the GMRES algorithm (in exact arithmetic). ***To cite this article: A.L. Bessoud, F. Krasucki, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***
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Résumé

Convergence q-superlinéaire de l'algorithme GMRES pour un multimatériau à liaison forte. On étudie un problème modèle non-classique de transmission, décrivant une multistructure composée de deux solides reliés par une jonction forte. En utilisant une méthode de décomposition de domaines, le problème se ramène à une équation définie sur l'interface, de la forme : $(\mathcal{I}-\mathcal{G})g=F$. On montre que $\mathcal{G}$ est compact de classe de Carleman C_s, et on en déduit la convergence q-superlinéaire de l'algorithme GMRES. ***Pour citer cet article : A.L. Bessoud, F. Krasucki, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***
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1. Position of the problem

For the sake of simplicity we shall consider a model problem, such as the heat conduction through two bodies, bonded by a thin strong layer whose thermal properties are large with respect to those of the two bodies. The same results hold for other situations, e.g. for linear elasticity or linear plates theories. With an asymptotic analysis, similar to that in [2,9,7], the thin strong layer 'disappears' from a geometrical point of view, but remains through a surface energy which gives rise to a non-classical transmission condition. In order to state this limit problem, we denote by Ω^+ and Ω^- the two bodies, which we assume to be open connected subsets of $\mathbb{R}^3$ with Lipschitz-continuous boundaries $\partial\Omega^+$ and $\partial\Omega^-$ and we assume that $S=\partial\Omega^+\cap\partial\Omega^-$ is a non-empty regular surface of positive measure. Let Ω be the interior of $(\overline{\Omega^+\cup\Omega^-})$, with boundary $\partial\Omega$. We assume that the following boundary conditions of mixed type hold: the temperature is zero on $\Gamma_0^-\cup\Gamma_0^+$ and the heat flux is given on the part Γ_φ^+ with $\Gamma_0^-=\partial\Omega^-\cap\partial\Omega$,

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$\Gamma_0^+ \cup \Gamma_\varphi^+ = \partial\Omega^+ \cap \partial\Omega$ and $\partial S \subset \Gamma_0^- \cup \Gamma_0^+$. For a function w defined on Ω, let w^+ and w^- denote the restrictions of w to Ω^+ and Ω^-, respectively. Let us define the space V:

$$V = \{v = (v^+, v^-);\ v^+ \in H^1(\Omega^+),\ v^- \in H^1(\Omega^-),\ v^+_{|S} = v^-_{|S} \in H^1_0(S) \text{ and } v^\pm = 0 \text{ on } \Gamma_0^\pm\} \tag{1}$$

whose associate norm is

$$\|v\|_V^2 = \|\nabla v^+\|^2_{L^2(\Omega^+)} + \|\nabla v^-\|^2_{L^2(\Omega^-)} + \|\nabla_\tau v\|^2_{L^2(S)},$$

where ∇_τ denotes the surface gradient operator. The problem can then be stated as follows:

$$\begin{cases} \text{Find } u \in V \text{ so that} \\ J(u) \leqslant J(v), \quad \forall v \in V, \end{cases} \tag{2}$$

where

$$J(v) = \frac{1}{2}a^+(v^+, v^+) + \frac{1}{2}a^-(v^-, v^-) + \frac{1}{2}a^s(v, v) - L^+(v^+) - L^-(v^-), \tag{3}$$

$$a^\pm(v^\pm, v^\pm) = \int_{\Omega^\pm} K^\pm \nabla v^\pm \nabla v^\pm \, \mathrm{d}\Omega, \tag{4}$$

$$a^s(v, v) = \int_S K^s \nabla_\tau v \nabla_\tau v \, \mathrm{d}S, \tag{5}$$

$$L^+(v^+) = \int_{\Omega^+} f^+ v^+ \, \mathrm{d}\Omega - \int_{\Gamma_\varphi^+} \varphi^+ v \, \mathrm{d}\Gamma; \qquad L^-(v^-) = \int_{\Omega^-} f^- v^- \, \mathrm{d}\Omega. \tag{6}$$

Under standard assumptions on the regularity of the loading and the positivity of the coefficients $K^\pm$ and K^s, we can easily prove the existence and the uniqueness of a solution to this problem.

Let us remark that the transmission conditions on S are $u^+ = u^-$ and the non-classical one:

$$K^+ \frac{\partial u^+}{\partial n^+} + K^- \frac{\partial u^-}{\partial n^-} = K^s \Delta_\tau u. \tag{7}$$

2. Multi-domains formulation

We cannot apply directly the iterative methods used for multi-domains to take into account the transmission condition (7) on the strong interface S. We propose a suitable domain decomposition method that reduces the problem to an integral equation on S. As in e.g. [6], we solve at first a classical non-homogeneous Dirichlet problem or a mixed problem on each subdomain with a boundary datum g on the interface S. To this end, let $g \in H^1_0(S)$ be given and define the spaces $V_g^+ = \{v \in H^1(\Omega^+);\ v = g \text{ on } S,\ v = 0 \text{ on } \Gamma_0^+\}$ and $V_g^- = \{v \in H^1(\Omega^-);\ v = g \text{ on } S,\ v = 0 \text{ on } \Gamma_0^-\}$. Let $u_g^\pm \in V_g^\pm$ be such that:

$$J^\pm(u_g^\pm) \leqslant J^\pm(v_g^\pm), \quad \forall v_g^\pm \in V_g^\pm, \tag{8}$$

where

$$J^\pm(v_g^\pm) = \frac{1}{2}a^\pm(v_g^\pm, v_g^\pm) - L^\pm(v_g^\pm). \tag{9}$$

Then $u_g = (u_g^+, u_g^-)$ is the solution to the global minimization problem (2) if and only if g is the (unique) solution to the following problem:

$$\begin{cases} \text{Find } g \in H^1_0(S), \text{ such that for all } g^* \in H^1_0(S) \\ a^s(g, g^*) + a^+(u_g^+, v_{g^*}^+) + a^-(u_g^-, v_{g^*}^-) - L^+(v_{g^*}^+) - L^-(v_{g^*}^-) = 0. \end{cases} \tag{10}$$

Using a Green formula, we can see that the previous problem can be written as:

$$\begin{cases} \text{Find } g \in H^1_0(S), \text{ such that for all } g^* \in H^1_0(S) \\ a^s(g, g^*) = \left\langle K^+ \frac{\partial u_g^+}{\partial n^+} + K^- \frac{\partial u_g^-}{\partial n^-}, g^* \right\rangle, \end{cases} \tag{11}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1/2}(S)$ and $H^{1/2}(S)\cap H_0^1(S)$. Let us denote by $G: H^{-1}(S)\to H_0^1(S)$ the solution operator corresponding to the Dirichlet problem for the form a^s. Hence g solves (11) if and only if:

$$g = G\left(K^+\frac{\partial u_g^+}{\partial n^+} + K^-\frac{\partial u_g^-}{\partial n^-}\right). \tag{12}$$

Since the maps $g\mapsto K^\pm\partial u_g^\pm/\partial n^\pm$ from $H_0^1(S)$ into $H^{-1/2}(S)$ are affine, we can write

$$G\left(K^+\frac{\partial u_g^+}{\partial n^+} + K^-\frac{\partial u_g^-}{\partial n^-}\right) = \mathcal{G}g + F,$$

where

$$\mathcal{G}g = G\left(K^+\left(\frac{\partial u_g^+}{\partial n^+} - \frac{\partial u_0^+}{\partial n^+}\right) + K^-\left(\frac{\partial u_g^-}{\partial n^-} - \frac{\partial u_0^-}{\partial n^-}\right)\right), \qquad F = G\left(K^+\frac{\partial u_0^-}{\partial n^+} + K^-\frac{\partial u_0^-}{\partial n^-}\right),$$

and $u_0^\pm$ denotes the solution of (9) for $g=0$.

The solution of problem (11) is hence reduced to solving the linear equation:

$$\mathcal{A}g \equiv (\mathcal{I}-\mathcal{G})g = F \quad \text{in } H_0^1(S). \tag{13}$$

The existence and uniqueness of the solution to problem (2) implies that (13) has a unique solution in $H_0^1(S)$.

In the GMRES method [11], the kth iterate u_k minimizes the norm of the residual $r_k = F - \mathcal{A}u_k$ over the affine space $u_0 + \mathcal{K}_k$ where $\mathcal{K}_k$ is the Krylov space generated by $r_0, \mathcal{A}r_0, \ldots, \mathcal{A}^{k-1}r_0$. As a consequence $\|r_k\| \leqslant \|p(\mathcal{A})\|\|r_0\|$, where $p(\mathcal{A})$ is any polynomial of degree k with $p(0)=1$. The rate of the decay of the spectrum of $\mathcal{G}$ can be used to choose a special sequence of polynomials in order to obtain convergence rate estimates.

Here, the main aim is to use the regularizing character of the operator $\mathcal{G}$ to obtain such convergence rate estimates for (13). For that, the following result is essential:

Theorem 2.1. *There exists t with $1/4 < t \leqslant 1/2$, such that the linear map $g\mapsto \mathcal{G}g$ is continuous from $H_0^1(S)$ to $H_0^1(S)\cap H^{1+t}(S)$.*

Proof. Since

$$K^+\frac{\partial u_g^+}{\partial n^+} + K^-\frac{\partial u_g^-}{\partial n^-} \in H^{-1/2}(S),$$

it follows from results of P. Grisvard [8] and by interpolation that

$$G\left(K^+\frac{\partial u_g^+}{\partial n^+} + K^-\frac{\partial u_g^-}{\partial n^-}\right) \in H^{1+t}(S)$$

for some $1/4 < t \leqslant 1/2$, the exact value of which depends on the regularity of the surface S. □

Corollary 2.2. *The linear map $g\mapsto \mathcal{G}g$ is compact of Carleman class C_s in $H_0^1(S)$ for every $s \geqslant 2/t$.*

Proof. Since S is contained in $\mathbb{R}^2$, the imbedding of $H^{1+t}(S)$ with $t > 1/4$ in $H^1(S)$ is compact of Carleman class C_s for every $s \geqslant 2/t$, cf. [1]. The result then follows from Theorem 2.1 and the property that the composition of a bounded operator and an operator of class C_s is an operator of class C_s. □

The classes C_s of compact operators in Hilbert spaces are studied in depth in ([5], Chapter XI), where are established completeness properties of the eigenfunctions generalizing earlier work of Carleman [4] for Hilbert–Schmidt operators.

These results allow to obtain the following result of convergence:

Theorem 2.3. *The unique solution of Eq.* (13) *can be approximated by the GMRES method, which converges q-superlinearly in exact arithmetic.*

Proof. The q-superlinearity convergence, i.e., the property that the sequence $\{r_n\}$ of the residuals satisfies: $\|r_{n+1}\|/\|r_n\| \to 0$, follows from Corollary 2.2 and a straightforward application of Theorem 1 of [10]. $\square$

The same reasoning based on the regularizing character of the analogous of the operator $\mathcal{G}$ can be used to prove the q-superlinear convergence of the GMRES method for others non-classical transmission conditions, for instance those appearing in the case of a bonded structure with a weak interface, as in [6].

With the same method, one can also prove the convergence of the GMRES algorithm for the finite element approximation (see [3] where suitable error estimates and numerical results are given also for linear elasticity problems).

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