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Dynamical Systems

Further reduction of normal forms of formal maps

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Abstract

Further reduction for classical normal forms of formal maps is considered in this note. Based on a recursive formula for computing the transformed map of a formal map under a near identity formal transformation, we develop the concepts of Nth order normal forms and infinite order normal forms for formal maps, and give some sufficient conditions for uniqueness of normal forms of formal maps. ***To cite this article: D. Wang et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Réduction supplémentaire des formes normales d'applications différentiables. La réduction supplémentaire des formes normales classiques d'applications formelle est étudiée dans cette note. En utilisant des formules récursives pour le calcul de l'application obtenue par une transformation formelle tangente à l'identité, nous développons la notion de formes normales d'ordre N et d'ordre infini pour les applications formelles, et nous donnons des conditions suffisantes pour l'unicité de ces formes normales. ***Pour citer cet article : D. Wang et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction

Since classical normal forms of formal maps (including $\mathcal{C}^\infty$ smooth maps) are, in general, not the simplest among formal conjugate class, further reduction of classical normal forms is necessary. We consider a formal map F with the origin as a fixed point:

$$F(x) = Ax + f_2(x) + f_3(x) + \cdots + f_r(x) + \cdots, \tag{1}$$

where $x \in \mathbb{C}^n$, A is an $n \times n$ complex constant matrix in Jordan canonical form and $f_k \in \mathbf{H}_n^k$, where $\mathbf{H}_n^k$ is the linear space of n-dimensional vector valued homogenous polynomials of degree k in n variables with coefficients in $\mathbb{C}$, $k = 2, 3, \ldots, r, \ldots$.

Under a near identity transformation

$$x = T(y) = y + \varphi(y) = y + \varphi_2(y) + \varphi_3(y) + \cdots + \varphi_k(y) + \cdots, \quad \varphi_k \in H_n^k, \tag{2}$$

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F can be changed to

$$H(y)=T^{-1}\circ F\circ T(y)=Ay+h_2(y)+h_3(y)+\cdots+h_k(y)+\cdots,\quad h_k\in H_n^k. \tag{3}$$

By the recursive formula given by Peng [3], we have

$$\begin{aligned}h_k={}&f_k-\big(\varphi_k(Ay)-A\varphi_k(y)\big)-\mathcal{S}_2^{(k-2)}[f_2,f_3,\dots,f_{k-1}](\varphi_2,\varphi_3,\dots,\varphi_{k-1})\\&-\mathcal{R}_2^{(k-2)}[f_2,\dots,f_{k-1},h_2,\dots,h_{k-1}](\varphi_2,\varphi_3,\dots,\varphi_{k-1})\end{aligned} \tag{4}$$

for $k=2,3,\dots,$ where

$$\begin{aligned}&\mathcal{S}_2^{(k-2)}[f_2,f_3,\dots,f_{k-1}](\varphi_2,\varphi_3,\dots,\varphi_{k-1})=\sum_{i=2}^{k-1}\big(D\varphi_i(Ay)f_{k+1-i}-Df_{k+1-i}(y)\varphi_i\big)\\&\quad+\sum_{r=2}^{[k/2]}\frac{1}{r!}\sum_{i=r}^{k-r}\sum_{\substack{l_1+\cdots+l_r=k-(i-r)\\2\leqslant l_1,\dots,l_r\leqslant k-(i-r)-2(r-1)}}\big\{D^r\varphi_i(Ay)f_{l_1}\cdots f_{l_r}-D^rf_i(y)\varphi_{l_1}\cdots\varphi_{l_r}\big\},\end{aligned}$$

$$\begin{aligned}&\mathcal{R}_2^{(k-2)}[f_2,\dots,f_{k-1},h_2,\dots,h_{k-1}](\varphi_2,\dots,\varphi_{k-1})=\sum_{i=2}^{k-1}\big\{D\varphi_i(Ay)(h_{k+1-i}-f_{k+1-i})\big\}\\&\quad+\sum_{r=2}^{[\frac{k}{2}]}\frac{1}{r!}\sum_{i=r}^{k-r}\sum_{\substack{l_1+\cdots+l_r=k-(i-r)\\2\leqslant l_1,\dots,l_r\leqslant k-(i-r)-2(r-1)}}\big\{D^r\varphi_i(Ay)h_{l_1}\cdots h_{l_r}-D^r\varphi_i(Ay)f_{l_1}\cdots f_{l_r}\big\},\end{aligned}$$

where the operator D is a Frechét derivative.

In Section 2, based on the formula (4), we develop a new framework of further reduction of normal forms for formal maps by combining a similar idea of Chen and Dora [1] and the method introduced by Kokubu, Oka and Wang [2].

2. Infinite order normal forms and unique normal forms

For the given formal map (1), we define a sequence of operators as follows. Let

$$\mathcal{L}_k^{(1)}=L_A^k:\mathbf{H}_n^k\longrightarrow\mathbf{H}_n^k,\quad \varphi_k\longmapsto\varphi_k(Ax)-A\varphi_k(x),\quad \varphi_k\in\mathbf{H}_n^k,\ \forall k\geqslant 2,\ k\in\mathbb{N}$$

and

$$\begin{aligned}&\mathcal{L}_k^{(m+1)}:\operatorname{Ker}\mathcal{L}_k^{(m)}\times\mathbf{H}_n^{k+m}\longrightarrow\mathbf{H}_n^{k+m},\quad\forall k\geqslant 2,\ k,m\in\mathbb{N}\\&\quad(\varphi_k,\dots,\varphi_{k+m-1},\varphi_{k+m})\longmapsto L_A^{k+m}(\varphi_{k+m})+\mathcal{S}_k^{(m)}[f_2,\dots,f_{m+1}](\varphi_k,\dots,\varphi_{k+m-1}),\end{aligned}$$

where

$$\begin{aligned}\mathcal{S}_k^{(m)}[f_2,\dots,f_{m+1}](\varphi_k,\dots,\varphi_{k+m-1})={}&\sum_{\substack{i+j=k+m+1\\i\geqslant k,\ j\geqslant 2}}\big(D\varphi_i(Ay)f_j-Df_j(y)\varphi_i\big)\\&+\sum_{r=2}^{m}\frac{1}{r!}\sum_{\substack{i=k\\i\geqslant r}}^{m+k-r}\sum_{\substack{l_1+\cdots+l_r=m+k-(i-r)\\2\leqslant l_1,\dots,l_r\leqslant m}}D^r\varphi_i(Ay)f_{l_1}\cdots f_{l_r}\\&-\sum_{r=2}^{[m/k]+1}\frac{1}{r!}\sum_{i=r}^{m+k-r(k-1)}\sum_{\substack{l_1+\cdots+l_r=m+k-(i-r)\\k\leqslant l_1,\dots,l_r\leqslant m+k-2}}D^rf_i(y)\varphi_{l_1}\cdots\varphi_{l_r}.\end{aligned}$$

It is obvious that for any integer $m\geqslant 1$, operator $\mathcal{L}_k^{(m)}$ depends on $A,f_2,\dots,f_m$ and can be denoted by $\mathcal{L}_k^{(m)}[A,f_2,\dots,f_m]$. We note that $\mathcal{L}_k^{(1)}$ and $\mathcal{L}_k^{(2)}$ are linear but when $m\geqslant 3$, $\mathcal{L}_k^{(m)}$ is in general nonlinear.

Definition 2.1. A formal map $H(x) = Ax + h_2(x) + h_3(x) + \cdots + h_r(x) + \cdots$, where $h_r \in \mathbf{H}_n^r$ for each integer $r \geqslant 2$, is called an Nth order normal form if $h_{1+i} \in C_{1+i}^i$, $1 \leqslant i \leqslant N$, and $h_{1+j} \in C_{1+j}^N$, $j \geqslant N+1$, $i, j \in \mathbb{N}$, where C_{1+j}^m is a complementary subspace to the maximal linear subspace R_{1+j}^m contained in the image of $\mathcal{L}_{2+j-m}^{(m)}$ in $\mathbf{H}_n^{1+j}$ for each integer $m \geqslant 1$ and $j \geqslant m$, where $\mathcal{L}_{2+j-m}^{(m)} = \mathcal{L}_{2+j-m}^{(m)}[A, h_2, \ldots, h_m]$. In addition, if $h_{1+i} \in C_{1+i}^i$, $\forall i \in \mathbb{N}$, then $H(x)$ is called an infinite order normal form.

In what follows, we assume that the choice of the complementary space C_{k+m}^m to R_{k+m}^m is fixed whenever R_{k+m}^m has been determined and if $\dim R_{k+m}^{m+1} = \dim R_{k+m}^m$, then we take $R_{k+m}^{m+1} = R_{k+m}^m$ and further $C_{k+m}^{m+1} = C_{k+m}^m$.

Definition 2.2. Assume that the rule of selecting complementary subspaces in the process of computing normal forms is fixed. Let the formal map $H(x)$ be an infinite order normal form. If there is no other infinite order normal form which is formally conjugate to $H(x)$, then $H(x)$ is called a unique normal form. For a given formal map $F(x)$, if $H(x)$ is an infinite order normal form of $F(x)$ and is a unique normal form, then $H(x)$ is called the unique normal form of $F(x)$.

Theorem 2.3. *Let $H(x) = Ax + h_2(x) + h_3(x) + \cdots + h_r(x) + \cdots$ be an infinite order normal form of* (1). *If* $\operatorname{Im} \mathcal{L}_2^{(k)}[A, h_2, \ldots, h_k] \cap C_{1+k}^k = \{0\}$ *for any integer $k \geqslant 1$, then $H(x)$ is a unique normal form of* (1).

In some cases, an Nth order normal form is already an infinite order normal form. It may be also a unique normal form.

Theorem 2.4. *If there exists $N \in \mathbb{N}$ such that* $\operatorname{Ker} \mathcal{L}_k^{(N)} = \{0\} \times \operatorname{Ker} \mathcal{L}_{k+1}^{(N-1)}$, *holds* for any integer $k \geqslant 2$, *then* $\operatorname{Im} \mathcal{L}_k^{(N+m)} = \operatorname{Im} \mathcal{L}_{k+m}^{(N)}$ for all $k \geqslant 2$, $m \geqslant 1$, $k, m \in \mathbb{N}$, *and hence the Nth order normal form is an infinite order normal form.*

Example 1. Consider one-dimensional map $F_0(x) = ax^r + \cdots$, with $a \neq 0$ and $r \geqslant 2$, $r \in \mathbb{N}$. It is easy to see that for any $m \geqslant 2$, $m \in \mathbb{N}$

$$\operatorname{Ker} \mathcal{L}_m^{(r)} = \{0\} \times \operatorname{Ker} \mathcal{L}_{m+1}^{(r-1)} \quad \text{and} \quad \operatorname{Im} \mathcal{L}_m^{(r)} = \mathbf{H}_1^{r+m-1}.$$

Then the rth order normal form of $F_0(x)$, $H(x) = ax^r$, is an infinite order normal form and is unique.

In what follows, we consider a special case and give a simpler condition for uniqueness of normal forms. Let $H(x) = Ax + h_\mu(x) + \cdots + h_{\mu+r}(x) + \cdots$, where A is diagonal and $h_\mu \neq 0$, $h_\mu \in C_\mu^1 (= C_\mu^{\mu-1})$, where C_μ^1 is some complementary space to $\operatorname{Im} L_A^\mu$. Then, for any $m \geqslant 2$, $m \in \mathbb{N}$, $\mathbf{H}_n^m = \operatorname{Im} L_A^m \oplus \operatorname{Ker} L_A^m$ and $h_\mu = h_{1,\mu} + h_{2,\mu}$ where $h_{1,\mu} \in \operatorname{Im} L_A^\mu$ and $h_{2,\mu} (\neq 0) \in \operatorname{Ker} L_A^\mu$. We define a linear operator $\mathcal{T}_m[h_{2,\mu}] : \operatorname{Ker} L_A^m \to \mathbf{H}_n^{\mu+m-1}$ by $\mathcal{T}_m[h_{2,\mu}](\varphi_m)(x) = D\varphi_m(Ax) h_{2,\mu}(x) - Dh_{2,\mu}(x)\varphi_m(x)$ for any integer $m \geqslant 2$.

Theorem 2.5. *Assume that a formal map has the following form $H(x) = Ax + h_\mu(x) + h_{\mu+1}(x) + \cdots + h_{\mu+r}(x) + \cdots$, where A is diagonal, $h_\mu \in C_\mu^1$ and $h_\mu \neq 0$. Let $h_{2,\mu}$ be the projection of h_μ onto* $\operatorname{Ker} L_A^\mu$. *If the following conditions are satisfied*:

(1) *For any integer $m \geqslant 2$, $m \neq \mu$,* $\operatorname{Ker} \mathcal{T}_m[h_{2,\mu}] = \{0\}$.

(2) *For any integer $m > \mu$, either* $\dim \operatorname{Ker} L_A^{\mu+m-1} = 0$ *or* $\dim \operatorname{Ker} L_A^{\mu+m-1} = \dim \operatorname{Ker} L_A^m$.

Then the μth order normal form of $H(x)$ is an infinite order normal form and is unique.

Remark 1. Under the conditions of Theorem 2.5, the unique normal form of $H(x)$ has the following form $H^{(\mu)} = Ax + h_\mu + h_{\mu+1}^{(\mu)} + \cdots + h_{2\mu-1}^{(\mu)}$. If, in addition, the condition(2) of Theorem 2.5 holds also for any integer $2 \leqslant m < \mu$, then the unique normal form is of the form $H^{(\mu)} = Ax + h_\mu + h_{2\mu-1}^{(\mu)}$.

Example 2. Consider one-dimensional map $F_1(x) = x + \sum_{k=\mu}^{\infty} a_k x^k (a_\mu \neq 0)$. It is easy to obtain that $\operatorname{Ker} L_A^k = \{d_k x^k | d_k \in \mathbb{R}\}$ where $k \geqslant 2,\ k \in \mathbb{N}$. Furthermore, $\operatorname{Ker} \mathcal{T}_m[a_\mu x^\mu] = \{0\}$ for any integer $m \geqslant 2,\ m \neq \mu$. So by Remark 1, the μth order normal form of $F_1(x)$,

$$H_1(x) = x + a_\mu x^\mu + b x^{2\mu-1},$$

is the unique normal form of $F_1(x)$.

Example 3. Consider another one-dimensional map $F_{-1}(x) = -x + \sum_{k=2\nu+1}^{\infty} a_k x^k\ (a_{2\nu+1} \neq 0)$. Similarly to Example 2, the conditions of Theorem 2.5 can be tested. Then based on Remark 1, the $(2\nu + 1)$th order normal form of $F_{-1}(x)$,

$$H_{-1}(x) = -x + a_{2\nu+1} x^{2\nu+1} + b x^{4\nu+1},$$

is the unique normal form of $F_{-1}(x)$.

Example 4. Consider the formal map with generalized Neimark–Sacker singularity

$$F_i(z) = \mathrm{e}^{\mathrm{i}\theta} z + c_m z^{m+1} \bar{z}^m + \sum_{k=2m+2}^{\infty} \sum_{j=0}^{k} c_{j,k-j} z^j \bar{z}^{k-j}, \quad c_m \neq 0,\ c_m \in \mathbb{C},$$

where θ satisfies $\mathrm{e}^{\mathrm{i}k\theta} \neq 1,\ \forall k \in \mathbb{N}$ and $c_{j,k-j},\ z \in \mathbb{C}$, $\bar{z}$ is the conjugate of z and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. Let $c_m \mathrm{e}^{-\mathrm{i}\theta} = a_m + \mathrm{i} b_m$, where $a_m, b_m \in \mathbb{R}$. If $a_m \neq 0$, then the conditions of Theorem 2.5 hold. Note that

$$\operatorname{Im} \mathcal{L}_{2m+1}^{(2m+1)} = \left\{\gamma_j z^j \bar{z}^{4m+1-j} \mid \gamma_j \in \mathbb{C},\ j = 0, \ldots, 4m+1,\ j \neq 2m+1\right\} \oplus \left\{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta} \beta z^{2m+1} \bar{z}^{2m} \mid \beta \in \mathbb{R}\right\},$$

so we can choose $C_{4m+1}^{2m+1} = \{\tilde{a}_{2m} \mathrm{e}^{\mathrm{i}\theta} z^{2m+1} \bar{z}^{2m} \mid \tilde{a}_{2m} \in \mathbb{R}\}$. Then by Remark 1, the $(2m + 1)$th order normal form of $F_i(z)$,

$$H_i(z) = \mathrm{e}^{\mathrm{i}\theta}\left[z + (a_m + \mathrm{i} b_m) z^{m+1} \bar{z}^m + \tilde{a}_{2m} z^{2m+1} \bar{z}^{2m}\right],$$

is the unique normal form of $F_i(z)$, where $\tilde{a}_{2m}$ is a real number uniquely determined by $F_i(z)$.

For the details of proofs and algorithms for computing coefficients of unique normal forms for above examples, we refer to [4].

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