



Differential Geometry

The isotropy lattice of a lifted action

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Received 17 November 2005; accepted after revision 15 May 2006

Presented by Étienne Ghys

Abstract

We obtain a characterization of the isotropy lattice for the lifted action of a Lie group G on TM and T^*M based only on the knowledge of G and its action on M . Some applications to symplectic geometry are also shown. **To cite this article:** *M. Rodríguez-Olmos, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Résumé

Le réseau d'isotropie d'une action induite (sur le fibré (co-)tangent). Nous obtenons une caractérisation du réseau d'isotropie de l'action induite d'un groupe de Lie G sur TM et T^*M basée uniquement sur la connaissance de G et de son action sur M . Quelques applications en géométrie symplectique sont également données. **Pour citer cet article :** *M. Rodríguez-Olmos, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Version française abrégée

Soit $G \times M \rightarrow M$ une action propre et différentiable d'un groupe de Lie G sur une variété lisse, paracompacte, connexe et de dimension finie. Cette action induit des actions de G sur TM et T^*M également propres et différentiables. Cette Note étudie quelques propriétés de ces actions induites, pouvant être exprimées directement à partir de l'action initiale de G sur M et de la représentation (co-)adjointe de G .

Pour tout $x \in M$, on note G_x le groupe d'isotropie de x . Par la théorie générale des actions propres, ce groupe est compact (voir [1]). Le *réseau d'isotropie* pour l'action de G sur M est l'ensemble $I(G, M) = \{(H) : \exists x \in M \text{ t.q. } G_x = H\}$, où (H) désigne la classe de conjugaison d'un sous-groupe compact $H \subset G$. Le *type d'orbite* (H) est le sous-espace $M_{(H)} = \{x \in M : G_x \in (H)\}$. De manière analogue, nous pouvons définir $I(G, X)$ et $X_{(H)}$ pour chaque sous-espace topologique X de M . Ces deux objets sont très importants dans l'étude des stratifications des espaces d'orbites M/G et X/G (voir [1,3,5]). Dans plusieurs situations de géométrie symplectique, il est nécessaire de connaître le réseau d'isotropie $I(G, TM)$ associé à l'action induite, qui est en général plus compliqué que $I(G, M)$. Dans ce genre de situations, l'action induite est habituellement calculée explicitement, puis le réseau $I(G, TM)$ en est déduit. Par contre, $I(G, M)$ est presque toujours connu ou très simple. Notre résultat principal (Théorème 0.1) est

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une caractérisation de $I(G, TM)$ en termes de $I(G, M)$ et de la représentation adjointe de G (qui est indépendante de M).

Pour chaque sous-groupe compact $H \subset G$ d’algèbre de Lie \mathfrak{h} , l’annihilateur $\text{ann } \mathfrak{h} \subset \mathfrak{g}^*$ est un H -espace. On définit un ordre partiel dans $I(G, M)$ par $(H_1) \leq (H_2)$ si un représentant de (H_1) est conjugué à un sous-groupe d’un représentant de (H_2) . Nous montrons le théorème suivant :

Théorème 0.1. *Soit L un sous-groupe de G . La classe (L) appartient à $I(G, TM)$ si et seulement si il existe $(H_1), (H_2) \in I(G, M)$ et $(K) \in I(H_2, \text{ann } \mathfrak{h}_2)$ tels que $(H_1) \leq (H_2)$ et $L = H_1 \cap K$.*

Applications à la géométrie symplectique

L’action induite de G sur le fibré cotangent $\tau : T^*M \rightarrow M$ est hamiltonienne, l’application moment équivariante étant $\mathbf{J} : T^*M \rightarrow \mathfrak{g}^*$ donnée par $\langle \mathbf{J}(\alpha_x), \xi \rangle = \langle \alpha_x, \xi_M(x) \rangle$, pour $\alpha_x \in T_x^*M$, $\xi \in \mathfrak{g}$ et ξ_M le champ de vecteurs engendré par ξ sur M . L’ensemble $\mathbf{J}^{-1}(\mu)$ est G -invariant lorsque μ est une valeur complètement isotropique (i.e. vérifiant $G_\mu = G$ pour la représentation coadjointe de G). On peut définir le μ -réseau de M et la μ -fermeture d’un type d’orbite $M_{(H)}$, respectivement comme

$$I^\mu(G, M) = \{ (H) \in I(G, M) : \mu \in \text{ann } \mathfrak{h} \} \quad \text{et}$$

$$\text{cl}^\mu(M_{(H)}) = \left\{ \coprod_{(K)} M_{(K)} : (K) \in I^\mu(G, M) \text{ et } (K) \geq (H) \right\},$$

où $(H) \in I^\mu(G, M)$. Finalement, le choix d’une métrique riemannienne sur M (pour laquelle l’action de G sur M soit isométrique), et d’un potentiel G -invariant $V \in C^\infty(M)$, permet de définir sur T^*M un système hamiltonien G -invariant par le processus standard. Ce système est appelé *système mécanique simple*, et son hamiltonien est de la forme « énergie cinétique plus énergie potentielle ». Il est possible de montrer que les équilibres relatifs pour ce système se trouvent dans l’ensemble $\mathcal{S} = \{ \alpha_x \in T^*M : \alpha_x = \mathbb{F}\mathbb{L}(\xi_M(x)), \forall \xi \in \mathfrak{g}, \forall x \in M \}$, où $\mathbb{F}\mathbb{L} : TM \rightarrow T^*M$ est l’isomorphisme équivariant de Legendre induit par la métrique (voir [2]). Appelons \mathcal{S} l’ensemble des équilibres relatifs possibles. L’application du Théorème 0.1 et de la théorie générale des espaces stratifiés impliquent la propriété suivante :

Proposition 0.2. *Soit $\mu \in \mathfrak{g}^*$ une valeur complètement isotropique pour l’application moment de l’action induite de G sur T^*M . Il vient :*

- (i) $I(G, \mathbf{J}^{-1}(0)) = I(G, M)$ et pour tout $(H) \in I(G, M)$, $\tau((\mathbf{J}^{-1}(0))_{(H)}) = \overline{M_{(H)}}$.
- (ii) $I(G, \mathbf{J}^{-1}(\mu)) = I^\mu(G, M)$ et pour tout $(H) \in I^\mu(G, M)$, $\tau((\mathbf{J}^{-1}(\mu))_{(H)}) = \text{cl}^\mu(M_{(H)})$.
- (iii) *Pour un système mécanique simple G -invariant sur M , le réseau d’isotropie de l’ensemble des équilibres relatifs possibles est*

$$\coprod_{(H) \in I(G, M)} G \cdot I(H, \text{ann } \mathfrak{h}),$$

où $G \cdot I(H, \text{ann } \mathfrak{h})$ désigne l’ensemble de classes de conjugaison sous G d’éléments de $I(H, \text{ann } \mathfrak{h})$.

1. Introduction

In this Note we make some remarks about the geometry of (co-)tangent-lifted actions that seem to be unknown or not available in the literature. The problem considered is the following: if a Lie group G acts properly on the manifold M , this action is characterized by its isotropy lattice, which is the set of conjugacy classes of stabilizers of points of M , partially ordered by the relation of subconjugation. The knowledge of this lattice is an important tool that offers valuable information about the topology of the stratification of the quotient space M/G and its singularities. In many cases of geometrical and mechanical importance, one is interested in the study of the quotient space of a tangent (or cotangent) bundle by the lift of a proper group action on its base. In these cases, the relevant information is then obtained by the knowledge of the isotropy lattice for the (co-)tangent-lifted action. Now, since both the tangent bundle of a manifold M and the tangent-lifted action of a group G on TM are completely obtained from the geometry of M

and the G -action on it, the isotropy lattice for the lifted action should be obtainable from the isotropy lattice for the base action (supposed to be known) without the need of a direct study of the tangent-lifted action separately. Our main result, Theorem 2.1, gives a characterization of the isotropy lattice for the lifted action of G on TM once the isotropy lattice for the action on M and the adjoint representation of G are known. From this result, an algorithm to obtain every element in the lattice of the lifted action can be easily implemented. Also, we obtain several conditions relating stabilizers and orbit types of TM and M , and some applications to the study of level sets of the momentum map in symplectic geometry.

Throughout this Note M will denote a smooth, paracompact, connected and finite-dimensional manifold on which the finite-dimensional Lie group G acts smoothly and properly. This action induces in the usual way a proper action on the tangent bundle TM (respectively on T^*M) which is also smooth and proper.

Definition 1.1. If $H \subseteq G$ is a subgroup of G and $x \in M$, denote by (H) the conjugacy class of H in G and by G_x the stabilizer of x .

- (i) The set $I(G, M) = \{(H): \text{there is some } x \in M \text{ with } G_x = H\}$ is called the *isotropy lattice* for the G -action on M .
- (ii) For any $(H) \in I(G, M)$, the set $M_{(H)} = \{x \in M: G_x \in (H)\}$ is called an *orbit type* (of type H) of M .

Sometimes we will also consider conjugacy classes with respect to smaller subgroups of G . We will use the same symbol (H) for these new classes without explicitly saying it, unless this is not clear from the context. Also, it is clear from the definition the meaning of $I(G, X)$ and $X_{(H)}$ for any subspace X of M .

There is a partial order on an isotropy lattice as follows: if (H_1) and (H_2) are two different elements of $I(G, M)$ then $(H_1) < (H_2)$ if H_1 is conjugated to a proper subgroup of H_2 . This condition is easily checked to be independent of the representatives chosen. The following proposition collects a number of important properties of this action, which can be consulted, for example, in [1,3,4].

Proposition 1.2. For a G -action on M the following hold:

- (i) Every stabilizer group G_x is compact.
- (ii) There is a G -invariant metric on M .
- (iii) (*Tube Theorem*) Let \mathbf{S} be a G_x -invariant orthogonal complement to $\mathfrak{g} \cdot x$ in $T_x M$ with respect to some G -invariant Riemannian metric on M and $O \subset \mathbf{S}$ a small enough G_x -invariant open ball centered at the origin in \mathbf{S} . Then the space $G \times_{G_x} O$ obtained by quotienting $G \times O$ by the G_x -action $g' \cdot (g, s) = (gg'^{-1}, g' \cdot s)$ is G -diffeomorphic to a G -invariant open neighborhood U of the orbit $G \cdot x$ by the map $\phi([g, s]) = g \cdot \exp_x(s)$, where the G -action on the left-hand side is given by $g_1 \cdot [g_2, s] = [g_1 g_2, s]$.
- (iv) The connected components of each orbit type $M_{(H)}$ are embedded submanifolds of M , and $M_{(H)} \subset \overline{M_{(H')}} for every $(H') \leq (H)$.$
- (v) There is a minimal class (H_0) in $I(G, M)$ such that $(H_0) \leq (H)$ for every $(H) \in I(G, M)$.

Remark 1. Besides $I(G, M)$, we will consider the isotropy lattice for the H -representation on $\mathfrak{g}/\mathfrak{h}$ given by $h \cdot [\xi] = [\text{Ad}_h \xi]$ for a compact subgroup H . Note that this action is induced from the restriction to H of the adjoint representation of G on \mathfrak{g} , and that the dual of $\mathfrak{g}/\mathfrak{h}$ is isomorphic to the annihilator of \mathfrak{h} , $\text{ann } \mathfrak{h} \subset \mathfrak{g}^*$. Furthermore, since H is compact this isomorphism can be chosen H -equivariant with respect to the restricted coadjoint representation $H \times \text{ann } \mathfrak{h} \rightarrow \text{ann } \mathfrak{h}$ given by $h \cdot \mu = (\text{Ad}_h^*)^{-1} \mu$. Therefore, pairs of elements identified by this isomorphism have identical stabilizers and hence $I(H, \mathfrak{g}/\mathfrak{h}) = I(H, \text{ann } \mathfrak{h})$.

2. The main result

Theorem 2.1. Let G act on M and by tangent lifts on TM , and L be a subgroup of G . Then $(L) \in I(G, TM)$ if and only if there exist $(H_1), (H_2) \in I(G, M)$ and $(K) \in I(H_2, \text{ann } \mathfrak{h}_2)$ such that $(H_1) \leq (H_2)$ and $L = H_1 \cap K$.

Proof. We will fix once and for all a G -invariant Riemannian metric on M . Let $x \in M$ have stabilizer $H = G_x$, then we can form the H -invariant orthogonal splitting $T_x M = \mathfrak{g} \cdot x \oplus \mathbf{S}$ as in the Tube Theorem (item (iii)) in Proposi-

tion 1.2). Note that $\xi_M(x) = 0$ if and only if $\xi \in \mathfrak{h}$, and so there is an H -isomorphism $\psi : \mathfrak{g}/\mathfrak{h} \times \mathbf{S} \rightarrow T_x M$ given by $\psi([\xi], s) = \xi_M(x) + s$. H -equivariance is understood with respect to the induced linear action of H on $T_x M$ and to the diagonal action on $\mathfrak{g}/\mathfrak{h} \times \mathbf{S}$ given by $h \cdot ([\xi], s) = ([\text{Ad}_h \xi], h \cdot s)$.

By equivariance of the tangent bundle projection $\tau : TM \rightarrow M$, for any element $v_x \in TM$ such that $\tau(v_x) = x$ the tangent bundle and base isotropy groups are related by $G_{v_x} \subseteq H = G_x$. Furthermore, $G_{v_x} = H_{v_x}$, where on the left-hand side we consider the stabilizer for the full lifted action of G on TM and on the right-hand side the stabilizer for the induced linear representation of H on the vector space $T_x M$. If $\psi^{-1}(v_x) = ([\xi], s)$ then $G_{v_x} = H_{[\xi]} \cap H_s$, which means that the stabilizer of any tangent vector v_x over x is the intersection of two representatives of the lattices $I(H, \mathfrak{g}/\mathfrak{h})$ and $I(H, \mathbf{S})$. Conversely, any such intersection is the stabilizer of some vector in $T_x M$. Recall now from Remark 1 that $I(H, \mathfrak{g}/\mathfrak{h}) = I(H, \text{ann } \mathfrak{h})$. We need now to identify the elements of $I(H, \mathbf{S})$. For that, using the Tube Theorem choose elements $s \in O \subset \mathbf{S}$ and $g \in G$. Then

$$gH_s g^{-1} = G_{[g,s]} = G_{\phi([g,s])}. \quad (1)$$

Note that using every $g \in G$ and $s \in O$ the image of ϕ covers the full neighborhood $U \subset M$ in the Tube Theorem. Also, $x \in M_{(H)}$, and then by item (iv) in Proposition 1.2 we have that for any $(H') \in I(G, M)$, $U \cap M_{(H')} \neq \emptyset \Leftrightarrow (H') \leq (H)$. This, together with (1), means that the stabilizers for the linear H -representation on \mathbf{S} are conjugated to subgroups H' such that $(H') \in I(G, M)$ and $(H') \leq (H)$. Conversely, if $(H') \in I(G, M)$ satisfies $(H') \leq (H)$, then there is a representative H' of (H') with $H' \subset H$ and such that the H -conjugacy class (H') belongs to $I(H, \mathbf{S})$. Now, making $H = H_2$ and $H' = H_1$ the result follows. \square

Remark 2. We can obtain all the classes in $I(G, TM)$ with an easy algorithm. First, choose representatives of every class in $I(G, M)$ such that if $(H') \leq (H)$ the corresponding representatives satisfy $H' \subset H$. It is always possible to choose a complete set of representatives for all the classes of $I(G, M)$ in this way, and we will call them normal representatives. Let H_0 be the normal representative corresponding to the minimal class of $I(G, M)$ (see item (v) in Proposition 1.2). We will say that H_0 has depth zero. Any other normal representative H will have depth $n + 1$ if there is a normal representative H' with depth n such that (1) $H' \subset H$ and (2) there is no other normal representative H'' with $H' \subset H'' \subset H$.

To compute all the classes in $I(G, TM)$ we start by computing the classes of $I(H_0, \text{ann } \mathfrak{h}_0)$. Then, for any n and every normal representative H of depth n intersect the classes of $I(H, \text{ann } \mathfrak{h})$ with the H -class of every normal representative of depth $0, \dots, n - 1$ included in H . All the classes obtained after iterating in n in this way can be made G -classes by conjugating in G . After removing the repeated ones, we obtain all the elements of $I(G, TM)$.

Remark 3.

- (i) Note that the choice of a G -invariant metric on M (always available by item (ii) in Proposition 1.2) provides a G -bundle isomorphism $\mathbb{F}L : TM \rightarrow T^*M$. Consequently, $I(G, TM) = I(G, T^*M)$, and since $\mathbb{F}L$ covers the identity in M , $\tau((TM)_{(H)}) = \tau((T^*M)_{(H)})$ for any $(H) \in I(G, TM)$. Here we have denoted also by τ the cotangent bundle projection $T^*M \rightarrow M$.
- (ii) It easily follows from the proof of Theorem 2.1 that if G is Abelian, then $I(G, TM) = I(G, M)$. Furthermore, for any $(H) \in I(G, M)$, then $\tau((TM)_{(H)}) = \overline{M}_{(H)}$.

3. Applications in symplectic geometry

Using Remark 3 we can translate every result previously obtained for $I(G, TM)$ to $I(G, T^*M)$ where G acts on T^*M by cotangent lifts. Recall that this lifted action is Hamiltonian for the canonical symplectic structure ω on T^*M , and then it is natural to study the properties of $I(G, T^*M)$ from the point of view of symplectic geometry. In this section we make some remarks along these lines. Namely, we compute the restricted isotropy lattice for every G -invariant level set of the momentum map. The knowledge of the isotropy lattice of level sets of the momentum map is an important piece of information in the theory of singular reduction (see [3,5]). From the dynamics side we also obtain a restriction on the conjugacy classes of stabilizers for possible relative equilibria in an important class of Hamiltonian systems.

Recall that a cotangent-lifted action of G on (T^*M, ω) has an equivariant momentum map $\mathbf{J}: T^*M \rightarrow \mathfrak{g}^*$ defined by $\langle \mathbf{J}(\alpha_x), \xi \rangle = \langle \alpha_x, \xi_M(x) \rangle$, for every $\alpha_x \in T_x^*M$ and $\xi \in \mathfrak{g}$. This means that a necessary condition for $\alpha_x \in \mathbf{J}^{-1}(\mu)$ is $\mu \in \text{ann } \mathfrak{g}_x$. Let now $\mu \in \text{im } \mathbf{J} \subset \mathfrak{g}^*$ be a totally isotropic momentum value, i.e. $G_\mu = G$ for the coadjoint representation. Equivariance of \mathbf{J} implies that $\mathbf{J}^{-1}(\mu)$ is G -invariant. Note also that since μ is totally isotropic, if $\mu \in \text{ann } \mathfrak{g}_x$ the same is true for the Lie algebra of any other representative of (G_x) . We will define the μ -lattice of M as

$$I^\mu(G, M) = \{(H) \in I(G, M) : \mu \in \text{ann } \mathfrak{h}\}.$$

If $(H) \in I^\mu(G, M)$ then the μ -closure of $M_{(H)}$ is defined as

$$\text{cl}^\mu(M_{(H)}) = \left\{ \coprod_{(K)} M_{(K)} : (K) \in I^\mu(G, M) \text{ and } (K) \geq (H) \right\}.$$

Note that any nonempty μ -closure is G -invariant.

Recall also that a Hamiltonian system on (T^*M, ω) acted by G by cotangent lifts is called a (symmetric) ‘simple mechanical system’ if its Hamiltonian function is of the form $h(\alpha_x) = \frac{1}{2} \|\mathbb{F}\mathbb{L}^{-1}(\alpha_x)\|^2 + V(x)$, where V is a G -invariant function on M and the norm $\|\cdot\|$ and the Legendre map $\mathbb{F}\mathbb{L}$ (see Remark 3) are taken with respect to a given Riemannian metric for which the G -action on M is isometric. It is well known (see [2]) that if α_x is a relative equilibrium¹ for this Hamiltonian system, then it must be of the form $\alpha_x = \mathbb{F}\mathbb{L}(\xi_M(x))$ for some $\xi \in \mathfrak{g}$. We will then call the G -invariant subset of T^*M defined as $\{\mathbb{F}\mathbb{L}(\xi_M(x)) : x \in M \text{ and } \xi \in \mathfrak{g}\}$ the set of ‘possible relative equilibria’. Note that this set depends only on M and the G -action on it, and is the same for any simple mechanical system defined on T^*M . The knowledge of the restricted isotropy lattice for the set of possible equilibria on T^*M gives an estimate of the stabilizers that relative equilibria of simple mechanical systems on T^*M can have. This fact is of importance in the qualitative analysis of these systems, as for instance it can predict the existence or exclusions of certain types of equivariant bifurcations for any simple mechanical system defined on T^*M .

Proposition 3.1. *For the cotangent lift of G on T^*M , and if $\mu \in \mathfrak{g}^*$ is any totally isotropic momentum value, the following properties are satisfied:*

- (i) $I(G, \mathbf{J}^{-1}(0)) = I(G, M)$ and for any $(H) \in I(G, M)$, $\tau((\mathbf{J}^{-1}(0))_{(H)}) = \overline{M_{(H)}}$.
- (ii) $I(G, \mathbf{J}^{-1}(\mu)) = I^\mu(G, M)$ and for any $(H) \in I^\mu(G, M)$, $\tau((\mathbf{J}^{-1}(\mu))_{(H)}) = \text{cl}^\mu(M_{(H)})$.
- (iii) For any G -invariant simple mechanical system on M , the isotropy lattice of the set of possible relative equilibria is

$$\coprod_{(H) \in I(G, M)} G \cdot I(H, \text{ann } \mathfrak{h}),$$

where $G \cdot I(H, \text{ann } \mathfrak{h})$ is the set of conjugacy classes in G of representatives of elements of $I(H, \text{ann } \mathfrak{h})$.

Proof. (i) is clearly a particular case of (ii). To prove (ii), fix $x \in M$ with $G_x = H$. Recall from the definitions of \mathbf{J} , $\mathbb{F}\mathbb{L}$ and $\psi : \mathfrak{g}/\mathfrak{h} \times \mathbf{S} \rightarrow T_x^*M$ that if $\alpha_x \in \mathbf{J}^{-1}(\mu) \cap T_x^*M$ then $\alpha_x = (\mathbb{F}\mathbb{L} \circ \psi)([\xi], s)$, where $[\xi]$ is fixed by H . This implies that $G_{\alpha_x} = H_s$, which already appears as a stabilizer of some point of $M \cap \text{im}(\exp_x)$. Also since $H_s \subseteq H$ and $\mu \in \text{ann } \mathfrak{h}$, then $\mu \in \text{ann } \mathfrak{g}_{\alpha_x}$ as well. Applying this to every point $x \in M$ we find that $I(G, \mathbf{J}^{-1}(\mu)) = I^\mu(G, M)$. Also, the same reasoning shows that for each $(K) \in I^\mu(G, M)$ such that $(K) \geq (H)$, there is at least an element in $(\mathbf{J}^{-1}(\mu))_{(H)} \cap T_x^*M$ at every $x \in M_{(K)}$, which ends the proof of (ii).

For (iii), recall that by equivariance of the map $\mathbb{F}\mathbb{L}$, the isotropy lattice for the set of possible relative equilibria is the same as the isotropy lattice for the subset of T^*M given by the collection of infinitesimal generators for the G -action on M . Fixing again a point $x \in M$ with isotropy $H = G_x$, every infinitesimal generator at x is of the form $v_x = \xi_M(x)$ or equivalently $\psi([\xi], 0)$ and hence the H -class (G_{v_x}) is in $I(H, \text{ann } \mathfrak{h})$, by the proof of Theorem 2.1. Computing this lattice over each base point in $M_{(H)}$ generates $G \cdot I(H, \text{ann } \mathfrak{h})$, since any such base point has stabilizer conjugated to H . Finally, doing the same over each orbit type of M is equivalent to taking the reunion of all $G \cdot I(H, \text{ann } \mathfrak{h})$ for every $(H) \in I(G, M)$. \square

¹ A relative equilibrium in a Hamiltonian system is a point for which its Hamiltonian evolution lies inside a group orbit.

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