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Dynamical Systems

Mandelbrot-like sets in dynamical systems with no critical points

Antônio Endler, Jason A.C. Gallas

Instituto de Física, Universidade Federal do Rio Grande do Sul, 91501-970 Porto Alegre, Brazil

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Abstract

We report the discovery of an infinite quantity of Mandelbrot-like sets in the real parameter space of the Hénon map, a bidimensional diffeomorphism not obeying the Cauchy–Riemann conditions and having no critical points. For practical applications, this result shows to be possible to stabilize infinitely many complex phases by tuning real parameters only. ***To cite this article: A. Endler, J.A.C. Gallas, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Ensembles de Mandelbrot dans les systèmes dynamiques sans points critiques. Nous rapportons la découverte d'une quantité infinie d'ensembles de Mandelbrot dans l'espace des paramètres réels de la application d'Hénon, un difféomorphisme à deux dimension qui ne suit pas les conditions de Cauchy–Riemann et qui ne possède pas de points critiques. Pour des applications pratiques, nous montrons qu'il est possible de stabiliser une quantité infinie de phases complexes en ajustant seulement des paramètres réels. ***Pour citer cet article : A. Endler, J.A.C. Gallas, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

The Hénon map $H_{a,b}: X \to X$ defined by $H_{a,b}(x, y) \mapsto (a - x^2 + by,\ x)$ has been intensively studied in mathematics, physics and other disciplines [15]. But the perspectives prevailing in these fields are rather different. While mathematicians center investigations in the case where $X = \mathbb{C}^2$ aiming full generality, practical applications are essentially restricted to the case $X = \mathbb{R}^2$, slaved by requirements imposed by the physical interpretations attached to parameters and variables. The situation may be contrasted, for instance, by comparing recent work of Bonnot [4] and Bonatto et al. [3]. The parameter space of the Hénon map when $X = \mathbb{R}^2$ was first considered by El Hamouly and Mira [13,14] in this Journal. Subsequent works include [22,16,19,2,12,17,11,1,21,7,8,20,9,10] and many references therein.

The archetypal model of complex dynamics is the quadratic family $Q_c(z) \equiv c - z^2$ and the archetypal set is the Mandelbrot set $\mathcal{M}$, a subset of the c-plane given by $\mathcal{M} = \{c \mid \lim_{n\to\infty} Q_c^n(0) \nrightarrow \infty\}$, where $z = 0$ is the critical point of $Q_c(z)$ and $Q_c^n(z)$ denotes the nth composition of the map with itself [6,18]. Discrete dynamics in one complex variable is equivalent to a restricted class of two-dimensional maps involving two real variables and two functions

E-mail addresses: aendler@if.ufrgs.br (A. Endler), jgallas@if.ufrgs.br (J.A.C. Gallas).

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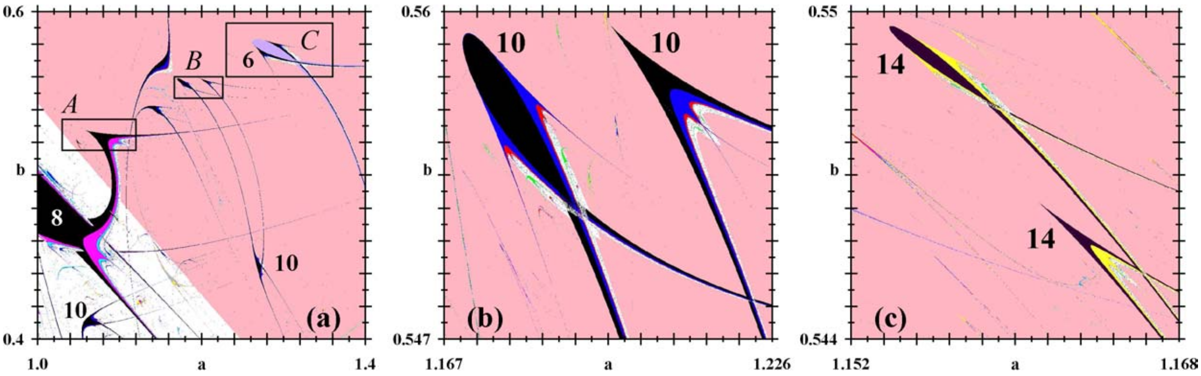

Fig. 1. Phase-diagrams illustrating two typical families of stability islands characterized by real-valued orbits.

that must obey Cauchy–Riemann conditions. But these conditions are not fulfilled by generic 2D dynamical systems routinely used as models of natural phenomena [15].

Our goal here is to report some remarkable numerical results obtained for the Hénon map when allowing for complex variables but still requiring parameters to be real, in a sort of middle ground between the aforementioned extreme situations. Mathematically, we investigate a quite different problem: a dynamical system having no critical points at all, the central players in the dynamics of complex functions.

The main novelty found is that by breaking the Cauchy–Riemann conditions one also breaks the known symmetries of the $\mathcal{M}$ set. Freed from the Cauchy–Riemann restriction one sees the abundant emergence of parameter domains with shapes that, while still somewhat resembling the $\mathcal{M}$ set, arise however with many variegated shapes, richer structures and with quite different metric properties. We refer generically to these domains as 'Mandelbrot-like' structures. Physically, they represent 'islands of stability', namely islands of Lyapunov stability with dynamics globally structurally stable. Here we focus on the qualitative description of these Mandelbrot-like structures which are typically present in multiparameter dynamical systems. Independently of how we move in parameter space, the global dynamical picture entails always a continuous and smooth interchange between infinite sets of complex and real stable orbits. Surprisingly, for real parameters, domains of *stable* real orbits are inexorably tied to domains of *stable* complex orbits in phase-diagrams and vice-versa.

The structuring of the parameter space is uncovered by studying for each parameter pair (a, b) the behavior of the iterates (x, y) and painting the point (a, b) with a specific color codifying the asymptotic behavior found. For instance, Fig. 1(a) illustrates the 'bare' parameter domains as usually obtained when considering *real* orbits only. The large pink background indicates parameters leading to unbounded orbits (divergence). The several colored structures correspond to periodic orbits, the period of the larger domains being indicated near to them. The white color codifies domains of 'chaos', namely either aperiodic orbits or orbits having period larger than 512. For better contrast, the main body of each stability island is shown in black. Whenever there is coexistence of orbits, we plotted the non-divergent orbit with basin of attraction of smallest volume in the rectangle $x \in [-2.5, 2.5]$ and $y \in [-10, 10]$, discretized on a 1024×1024 grid. Box B in Fig. 1(a) is magnified in Fig. 1(b). It displays two different types of singular structures which arise from a central body of period 10: a cuspidal and a rounded structure. These singularities may be described using the two normal forms of a cubic polynomial defined by the sign of the cubic term [5]. Fig. 1(c) shows a similar pair of singularities emanating from a central body of period 14 (not shown). Both structures seen in Figs. 1(b) and 1(c) appear frequently in experiments [3]. The structures in boxes A and C of Fig. 1(a) appear magnified in Figs. 2 and 3, where they now include the new decorations due to the *stable* complex orbits. Each individual figure displays 600×600 parameters.

The upper left corner in Fig. 2(a) shows an example of the new remarkable domains typically found near cuspidal singularities when stable complex orbits are also taken into account: the emergence of domains resembling 'cactus flowers', i.e. shapes resembling distorted Mandelbrot sets. The black cusp in Fig. 2(a) has period-8, the same period of the central yellow body of the cactus flower shown magnified in Fig. 2(b). Parameters from this body produce pairs of stable *complex conjugate* orbits, having the sum σ of the x coordinates of the orbit characterized by complex values. When changing parameters downwards near to the symmetry axis of Fig. 2(a) one finds a continuous conversion from a 8×2^n cascade of *complex* orbits into a 8×2^n cascade of *real* orbits. These cascades are interconnected by

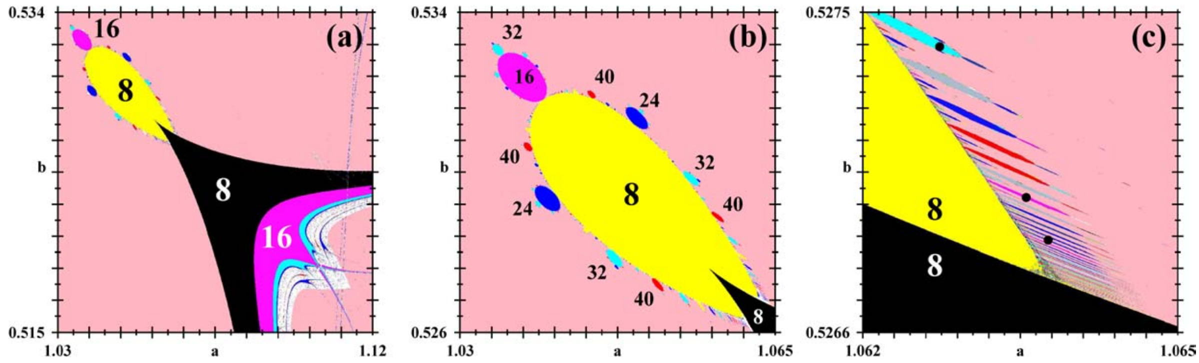

Fig. 2. Typical decoration produced by complex-valued orbits near a cuspidal singularity, here of period 8.

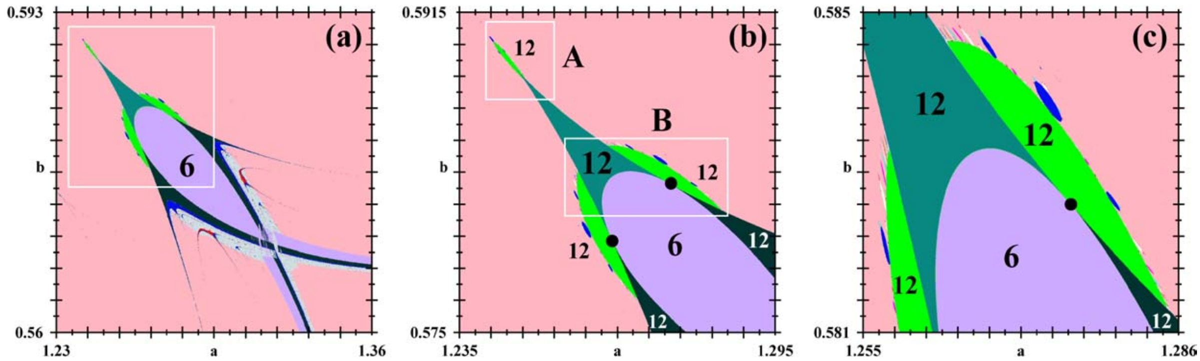

Fig. 3. Typical decoration produced by complex-valued orbits near the tip of a non-cuspidal singularity, here of period 6.

the two period-8 domains: the period-8 domain of the cactus of complex orbits and the period-8 cusp of real orbits. Numbers denote periods. Fig. 2(c) illustrates new phenomena found profusely: (i) the *accumulation* of self-similar cactus flowers toward the black domain characterized by real orbits, (ii) Mandelbrot-like structures orderly packed along *line segments*, not closed curves. To indicate the speed of convergence towards the accumulation point, black dots in Fig. 2(c) mark main bodies of periods 72, 120 and 168, from top to bottom.

Fig. 3(a) illustrates the striking structure found generically on the top of non-cuspidal singularities, making them look altogether similar to a 'sword-fish'. As before, the new domains seen are also due to stable complex orbits. They complement here the non-cuspidal structure inside box C in Fig. 1, due to real orbits of period 6 whose exact analytical expression was given in [7,8]. The non-cuspidal period-6 top forming the central nucleus is surrounded by five period-12 domains. There is a trivial pair of symmetrically placed cuspidal structures seen on the side of the central core. They correspond to doublings of the original period-6 orbits. The three green islands lying almost totally inside box B in Fig. 3(b) show the central bodies of non-trivial highly structured domains due to complex-valued orbits. The 'nose' along the symmetry axis arises from single complex orbits having a real value for the sum σ of their x orbital coordinates [9], i.e. formed by *complex conjugate* orbital points. The structuring found on the cuspidal top of the nose inside box A (singularity of the $+1$ eigenvalue locus) is analogous to that in Fig. 2(a), despite the complex nature of the orbits here. Identical structuring exist on the tip of an analogous period-6 bow-tie cusp reported analytically in [8] (orbits up to period 11 in [10]). The pair of black dots mark bifurcations points of codimension two where *four* stability islands meet. The situation around one of these remarkable points is shown in detail in Fig. 3(c). Parameters taken from the light-green period-12 domain lying above the curve containing the black dot in Fig. 3(c) lead to pairs of stable *conjugate* complex orbits.

We believe our investigation to be accurate albeit not rigorous, to shed new light on a matter which seemed already well exploited, and to be generic beyond the example selected. The results here pose an intriguing question: if not critical points, what exactly is behind self-similar Mandelbrot-like sets more generically? The isomorphism observed recently [3] between the parameter space of the Hénon map and that of a popular CO_2 laser model makes us believe that the singularities reported here might be *directly* observable in the laboratory in a not too distant future. Additional results will be presented elsewhere.

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