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Analytic Geometry

On the compactification of hyperconcave ends

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Abstract

We find a class of manifolds whose 'pseudoconcave holes' can be filled in, even in dimension two. ***To cite this article: G. Marinescu, T.-C. Dinh, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Compactification de bouts hyperconcaves. On étudie une classe de variétés dont les bouts strictement pseudoconcaves peuvent être compactifiés, même en dimension deux. ***Pour citer cet article : G. Marinescu, T.-C. Dinh, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

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Il est bien connu [16] qu'on peut compactifier les bouts strictement pseudoconcaves des variétés de dimension supérieure ou égale à trois. En revanche, en dimension deux, il existe des contre-examples très intéressants [9,1,16]. Dans ce papier on montre que les bouts hyperconcaves sont compactifiables même en dimension deux. On dit qu'une variété X a des bouts hyperconcaves si elle possède une fonction $\varphi : X \to (-\infty, a)$, $a \in \mathbb{R} \cup \{\infty\}$, propre, lisse et strictement plurisousharmonique sur un ensemble $\{\varphi < b\}$, où $b \leqslant a$.

Théorème 0.1. *Toute variété X avec des bouts hyperconcaves admet une compactification, i.e., il existe un espace complexe $\widehat{X}$ tel que X est (biholomorphe à) un ouvert de $\widehat{X}$ et pour tout $d < a$, $(\widehat{X} \setminus X) \cup \{\varphi \leqslant d\}$ est compact. Si φ est strictement plurisousharmonique sur X on peut prendre pour $\widehat{X}$ un espace de Stein à singularités isolées.*

En outre, on trouve des conditions suffisantes naturelles afin que la compactification se fasse en ajoutant un seul point à chaque bout.

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Théorème 0.2. *Soient X une variété avec bouts hyperconcaves et $\widehat{X}$ une compactification lisse. Supposons que X a un recouvrement par des ouverts de Zariski dont les revêtements universels sont de Stein. Alors $\widehat{X} \setminus X$ consiste en un ensemble fini D' et un ensemble analytique exceptionel qui se contracte sur un ensemble fini D. Chaque composante connexe de X_c, pour c assez petit, peut être compactifié analytiquement par un seul point de $D' \cup D$. Si X admet un revêtement de Stein alors $D' = \varnothing$ et D coincide avec le lieu singulier de la réduction de Remmert de $\widehat{X}$.*

Cela permet de donner une nouvelle preuve du théorème de Siu–Yau [18] et de généraliser les théorèmes de Nadel [14] en dimension deux. Nous utilisons la technique L^2 afin de construire des fonctions holomorphes permettant d'immerger les bouts hyperconcaves dans un $\mathbb{C}^N$ et d'utiliser le théorème d'Hartogs pour les compactifier. Pour le théorème 0.2, nous utilisons un théorème classique de Wermer.

1. Introduction

We will be concerned with the following class of manifolds.

Definition 1. A complex manifold X with $\dim X \geqslant 2$ is said to be a *strongly pseudoconcave end* if there exists a proper, smooth function $\varphi : X \to (e, a)$, $a \in \mathbb{R} \cup \{+\infty\}$, which is strongly plurisubharmonic on a set of the form $\{\varphi < b\}$, $b \leqslant a$. If $e = -\infty$, X is called a *hyperconcave end*. For $d < c < a$ we set $X_c = \{\varphi < c\}$ and $X_d^c = \{d < \varphi < c\}$. We call φ *exhaustion function*.

We say that a strongly pseudoconcave end can be *compactified* or *filled in* if there exists a complex space $\widehat{X}$ such that X is (biholomorphic to) an open set in $\widehat{X}$ and for any $d < a$, $(\widehat{X} \setminus X) \cup \{\varphi \leqslant d\}$ is a compact set. We will call $\widehat{X}$ the *completion* of X.

By a theorem of Rossi [16, Th. 3, p. 245] and Andreotti–Siu [1, Prop. 3.2] any strongly pseudoconcave end X can be compactified, provided $\dim X \geqslant 3$. This is no longer true if $\dim X = 2$, as shown in a counterexample of Grauert, Andreotti–Siu and Rossi [9,1,16]. Our goal is to compactify the hyperconcave ends also in dimension two.

Theorem 2. *Any hyperconcave end X can be compactified. Moreover, if φ is strongly plurisubharmonic on the whole X, the completion $\widehat{X}$ can be chosen a normal Stein space with at worst isolated singularities.*

The motivation for the study of the compactification of hyperconcave ends comes from the theory of complex-analytic compactification of quotients $X = \mathbb{B}^n/\Gamma$ of the unit ball in $\mathbb{C}^n$, $n \geqslant 2$ by arithmetic groups Γ. The Satake–Baily–Borel compactification $\widehat{X}$ of $X = \mathbb{B}^n/\Gamma$ is obtained by adding a finite set of points which are isolated singularities. Siu–Yau's theorem gives a differential geometric proof of this fact, first by proving that X has hyperconcave ends and then showing it can be compactified by adding finitely many points. Our next result is to find sufficient conditions for a manifold with hyperconcave ends to be analytically compactified by adding one point at each end. This also yields a complex analytic proof of the second step of Siu–Yau's theorem (see Corollary 10).

Theorem 3. *Let X be a hyperconcave end and let $\widehat{X}$ be a smooth completion of X. Assume that X can be covered by Zariski-open sets which are uniformized by Stein manifolds. Then $\widehat{X} \setminus X$ is the union of a finite set D' and an exceptional analytic set which can be blown down to a finite set D. Each connected component of X_c, for sufficiently small c, can be analytically compactified by one point from $D' \cup D$. If X itself has a Stein cover, $D' = \varnothing$ and D consists of the singular set of the Remmert reduction of $\widehat{X}$.*

Theorem 3 affords the extension to dimension two of Nadel's theorems [14]. If in Definition 1 we may take $a = +\infty$ and φ bounded from above, the manifold X is said to be *hyper* 1*-concave*.

Corollary 4. *Let X be a connected manifold of dimension $n \geqslant 2$. The following conditions are necessary and sufficient for X to be a quasiprojective manifold which can be compactified to a Moishezon space by adding finitely many points*: (i) *X is hyper* 1*-concave,* (ii) *X admits a line bundle E such that the ring $\bigoplus_{k>0} H^0(X, E^k)$ separates points and gives local coordinates, and* (iii) *X can be covered by Zariski-open sets which can be uniformized by Stein manifolds. If X has a Stein cover, one adds only singular points.*

Theorem 4 yields, in dimension two, a stronger version of Nadel–Tsuji theorem [15] together with a completely complex-analytic proof of the compactification of arithmetic quotients. It answers also [13, Prob. 1] for the case $q = 0$.

A detailed version of this Note is available in [11]. See also [12] for related questions.

2. Proof of Theorem 2

The idea of proof is to analytically embed small strips $X_d^{d^*}$, for $d < d^*$ in a neighbourhood of minus infinity, into the difference of two concentric polydiscs. Then apply the Hartogs extension theorem to extend the image to an analytic set which will provide the compactification. To obtain the embedding we follow the strategy of Grauert and Kohn for the solution of the Levi problem. Namely, we solve the L^2 $\bar\partial$-Neumann problem for (0, 1)-forms (see [7]) on domains X_c with strongly pseudoconvex boundary $\{\varphi = c\}$ endowed with a complete metric at minus infinity. The function $\chi = -\log(-\varphi)$ is smooth on X_0. We set $\omega = \sqrt{-1}\,\partial\bar\partial\chi = -\sqrt{-1}\,\partial\bar\partial\log(-\varphi)$. Note that $\partial\bar\partial\chi = \partial\bar\partial\varphi/(-\varphi) + (\partial\varphi\wedge\bar\partial\varphi)/\varphi^2$ and $(\partial\varphi\wedge\bar\partial\varphi)/\varphi^2 = \partial\chi\wedge\bar\partial\chi$. Since $\sqrt{-1}\,\partial\bar\partial\varphi/(-\varphi)$ represents a metric on X_0, we get the Donnelly–Fefferman condition:

$$|\partial\chi|_\omega \leqslant 1. \tag{1}$$

Since $\chi : X_0 \to \mathbb{R}$ is proper, (1) also ensures that ω is complete. Let $c < 0$ be a regular value of φ. The metric ω is complete at the pseudoconcave end of X_c and extends smoothly over the boundary bX_c.

We wish to derive the Poincaré inequality for (0, 1)-forms on X_c. For this goal we look first at the minus infinity end and use the Berndtsson–Siu trick [3,17]. Roughly speaking, it uses the negativity of the trivial line bundle, thus avoiding the problems raised by the control of the Ricci curvature of ω at $-\infty$. Let us denote by $\mathcal{C}_0^{0,q}(X_c)$ the space of smooth $(0, q)$-forms with compact support in X_c. Let $\vartheta = -*\partial*$ be the formal adjoint of $\bar\partial$ with respect to the scalar product $(u, v) = \int_{X_c}\langle u, v\rangle\,\mathrm{d}V_\omega$, where $\langle u, v\rangle = \langle u, v\rangle_\omega$ and $\mathrm{d}V_\omega = \omega^n/n!$.

Lemma 5. *For any $v \in \mathcal{C}_0^{0,1}(X_c)$ we have $\|v\|^2 \leqslant 8(\|\bar\partial v\|^2 + \|\vartheta v\|^2)$.*

Proof. On the trivial bundle $E = X_c \times \mathbb{C}$ we introduce the auxiliary Hermitian metric $\mathrm{e}^{\chi/2}$. Let be ϑ_χ the formal adjoint of $\bar\partial$ with respect to the scalar product $(u, v)_\chi = \int_{X_c}\langle u, v\rangle\,\mathrm{e}^{\chi/2}\,\mathrm{d}V_\omega$. Then $\vartheta_\chi = \mathrm{e}^{-\chi/2}\vartheta\,\mathrm{e}^{\chi/2}$. We apply the Bochner–Kodaira–Nakano formula for $u \in \mathcal{C}_0^{0,1}(X_c)$:

$$\int_{X_c}\big\langle\big[\sqrt{-1}\,\partial\bar\partial(-\chi/2), \Lambda_\omega\big]u, u\big\rangle\mathrm{e}^{\chi/2}\,\mathrm{d}V_\omega \leqslant \int_{X_c}\big(|\bar\partial u|^2 + |\vartheta_\chi u|^2\big)\,\mathrm{e}^{\chi/2}\,\mathrm{d}V_\omega, \tag{2}$$

where Λ_ω represents the contraction with ω and $[A, B] = AB - (-1)^{\deg A\cdot\deg B}BA$ is the graded commutator of the operators A, B. The idea is to substitute $v = u\,\mathrm{e}^{\chi/4}$. It is readily seen that

$$|\bar\partial u|^2\,\mathrm{e}^{\chi/2} \leqslant 2|\bar\partial v|^2 + \frac{1}{8}|\bar\partial\chi|^2|v|^2, \qquad |\vartheta_\chi u|^2\,\mathrm{e}^{\chi/2} \leqslant 2|\vartheta v|^2 + \frac{1}{8}|\partial\chi|^2|v|^2. \tag{3}$$

Moreover $\langle[\sqrt{-1}\,\partial\bar\partial(-\chi/2), \Lambda_\omega]u, u\rangle\,\mathrm{e}^{\chi/2} = \langle[\sqrt{-1}\,\partial\bar\partial(-\chi/2), \Lambda_\omega]v, v\rangle$. In general, for a (p, q)-form α we have the identity $\langle[\omega, \Lambda_\omega]\alpha, \alpha\rangle = (p + q - n)|\alpha|^2$, where $n = \dim X$. Taking into account that $\omega = \sqrt{-1}\,\partial\bar\partial\chi$ and that v is a (0, 1)-form, we obtain $\langle[\sqrt{-1}\,\partial\bar\partial(-\chi/2), \Lambda_\omega]u, u\rangle\,\mathrm{e}^{\chi/2} = \frac{n-1}{2}|v|^2 \geqslant \frac{1}{2}|v|^2$. By (2), (3), (1),

$$\frac{1}{2}\int_{X_c}|v|^2\,\mathrm{d}V_\omega \leqslant 2\int_{X_c}\big(|\bar\partial v|^2 + |\vartheta v|^2\big)\,\mathrm{d}V_\omega + \frac{1}{4}\int_{X_c}|v|^2\,\mathrm{d}V_\omega. \tag{4}$$

This immediately implies Lemma 5 for elements $v \in \mathcal{C}_0^{0,1}(X_c)$. □

Let $\eta : (-\infty, 0) \to \mathbb{R}$ be a smooth function such that $\eta(t) = 0$ on $(-\infty, -2]$, $\eta'(t) > 0$, $\eta''(t) > 0$ on $(-2, 0)$. Let us introduce the scalar product $(u, v)_{\eta(\varphi)} = \int_{X_c}\langle u, v\rangle\,\mathrm{e}^{-\eta(\varphi)}\,\mathrm{d}V_\omega$, the corresponding norm $\|\cdot\|_{\eta(\varphi)}$ and L^2 spaces, denoted $L_2^{0,q}(X_c, \eta(\varphi))$. Let $\mathcal{C}_0^{0,q}(\overline{X}_c)$ be the space of smooth $(0, q)$-forms with compact support in $\overline{X}_c$. Consider the maximal closed extension of $\bar\partial$ to $L_2^{0,q}(X_c, \eta(\varphi))$ and let $\bar\partial^*_{\eta(\varphi)}$ be its the Hilbert-space adjoint.

Lemma 6. *If η grows sufficiently fast, there exists a constant $C > 0$ such that*

$$\|u\|^2_{\eta(\varphi)} \leqslant C\left(\|\bar\partial u\|^2_{\eta(\varphi)} + \|\bar\partial^*_{\eta(\varphi)} u\|^2_{\eta(\varphi)} + \int_K |u|^2 \,\mathrm{e}^{-\eta(\varphi)}\,\mathrm{d}V_\omega\right), \tag{5}$$

for any $u \in \operatorname{Dom}\bar\partial \cap \operatorname{Dom}\bar\partial^*_{\eta(\varphi)} \subset L^{0,1}_2(X_c, \eta(\varphi))$, where $K = \{-3 \leqslant \varphi \leqslant -3/2\}$.

Proof. The Morrey–Kohn–Hörmander estimate [10, Th. 3.3.5] shows that there exists $R > 0$ such that for sufficiently growing η:

$$\|u\|^2_{\eta(\varphi)} \leqslant R\left(\|\bar\partial u\|^2_{\eta(\varphi)} + \|\bar\partial^*_{\eta(\varphi)} u\|^2_{\eta(\varphi)} + \int_{\{-3\leqslant\varphi\leqslant-3/2\}} |u|^2 \,\mathrm{e}^{-\eta(\varphi)}\,\mathrm{d}V_\omega\right), \tag{6}$$

for $u \in \operatorname{Dom}\bar\partial \cap \operatorname{Dom}\bar\partial^*_{\eta(\varphi)} \subset L^{0,1}_2(X_c, \eta(\varphi))$, $\operatorname{supp} u \subset \{-3 \leqslant \varphi\}$. Let $u \in \operatorname{Dom}\bar\partial \cap \operatorname{Dom}\bar\partial^*_{\eta(\varphi)} \subset L^{0,1}_2(X_c, \eta(\varphi))$. The density lemma of Andreotti–Vesentini shows that to prove (6) it suffices to consider smooth elements u compactly supported in $\overline{\Omega}$. We choose a cut-off function $\rho_1 \in \mathcal{C}^\infty(X_c)$ such that $\operatorname{supp}\rho_1 = \{-3 \leqslant \varphi\}$, $\rho_1 = 1$ on $\{-2 \leqslant \varphi\}$. Set $\rho_2 = 1 - \rho_1$. On $\operatorname{supp}\rho_2$, $\eta(\varphi)$ vanishes, therefore $\bar\partial^*_{\eta(\varphi)}(\rho_2 u) = \vartheta(\rho_2 u)$. Lemma 5 for $\rho_2 u$ gives $\|\rho_2 u\|^2_{\eta(\varphi)} \leqslant 8(\|\bar\partial(\rho_2 u)\|^2_{\eta(\varphi)} + \|\bar\partial^*_{\eta(\varphi)}(\rho_2 u)\|^2_{\eta(\varphi)})$. The latter estimate and (6) applied to $\rho_1 u$ together with standard inequalities deliver (5). □

In the sequel we fix a function η as in Lemma 6. Then the fundamental estimate (5) implies the solution of the L^2 $\bar\partial$-Neumann problem. Consider the complex of closed, densely defined operators $T = \bar\partial : L^{0,0}_2(X_c, \eta(\varphi)) \to L^{0,1}_2(X_c, \eta(\varphi))$, $S = \bar\partial : L^{0,1}_2(X_c, \eta(\varphi)) \to L^{0,2}_2(X_c, \eta(\varphi))$.

Theorem 7. *The operator T has closed range and* $\operatorname{Range}(T)$ *has finite codimension in* $\operatorname{Ker}(S)$. *If $f \in \operatorname{Range} T$, there is a unique solution $u \perp \operatorname{Ker} T$ of the equation $Tu = f$; if f is smooth in Ω so is u.*

By solving the $\bar\partial$-equation we construct peak functions at each point of bX_c.

Corollary 8. *Let $p \in bX_c$ and f be a holomorphic function on a neighbourhood of p such that $\{f = 0\} \cap \overline{X}_c = \{p\}$. Then for every m big enough, there is a function $g \in \mathcal{O}(X_c) \cap \mathcal{C}^\infty(\overline{X}_c \setminus \{p\})$, a smooth function Φ on a neighbourhood V of p and constants $a_1, \dots, a_{m-1}$ such that $g = f^{-m}(1 + a_{m-1}f + \cdots + a_1 f^{m-1}) + \Phi$ on $V \cap \Omega$. In particular, we have* $\lim_{z\to p} |g(z)| = \infty$.

Proceeding as in [1, Prop. 3.2] we show that $X^{d^*}_d$, for $c - \delta < d < d^* < c$ can be holomorphically embedded as submanifold of dimension $\geqslant 2$ in the difference of two concentric polydiscs in $\mathbb{C}^N$, for some N. By the Hartogs theorem we compactify X^c_d to a Stein space $\widehat{X}_c \subset \mathbb{C}^N$. The uniqueness of the Stein completion [1, Cor. 3.2] entails that $\widehat{X}_c$ does not depend on c, so letting $c \to -\infty$ we obtain the desired compactification $\widehat{X}$ of X.

Remark. Our method was to embed small strips $X^{d^*}_d$, for $c - \delta < d < d^* < c$ in $\mathbb{C}^N$ using holomorphic functions and apply the Hartogs phenomenon. One can produce easily holomorphic $(n, 0)$-forms on X_0 and an embedding $\Psi : X^{d^*}_d \to \mathbb{CP}^N$, using the standard L^2 estimates for $\bar\partial$ [5]. However, the global Hartogs or Harvey–Lawson phenomenon in $\mathbb{CP}^N$ is an open question [6, Problème 1]. Note that by pulling back $\Psi(X^{d^*}_d)$ to $\mathbb{C}^{N+1} \setminus \{0\}$ we obtain a noncompact manifold, so we cannot apply the known results from the euclidean space.

3. Compactification by adding finitely many points

The present section is devoted to proving sufficient conditions for the set $\widehat{X} \setminus X$ to be analytic. We begin with some preparations. We say that a complex manifold V satisfies the *Kontinuitätssatz* if for any smooth family of closed holomorphic discs $\bar\Delta_t$ in V indexed by $t \in [0, 1)$ such that $\bigcup b\Delta_t$ lies on a compact subset of V, then $\bigcup \bar\Delta_t$ lies on a compact subset of V. If V or its universal cover is Stein then V satisfies Kontinuitätssatz. A closed subset F of V is called *pseudoconcave* if $V \setminus F$ satisfies the local Kontinuitätssatz in V, i.e. for every $x \in F$ there is a neighbourhood W of x such that $W \setminus F$ satisfies the Kontinuitätssatz. Finite unions of pseudoconcave subsets and complex hypersurfaces are pseudoconcave.

We have the following proposition which implies Theorem 3.

Proposition 9. *Let $\widehat{\Omega}$ be a Stein space with isolated singularities S and K a completely pluripolar compact subset of $\widehat{\Omega}$ which contains S. Assume that $\Omega = \widehat{\Omega} \setminus K$ can be covered by Zariski-open sets which satisfy the local Kontinuitätssatz in $\widehat{\Omega} \setminus S$. Then K is a finite set. If $\Omega = \widehat{\Omega} \setminus K$ has a Stein cover, $K = S$.*

Proof. We can suppose that $\widehat{\Omega}$ is a subvariety of a complex space $\mathbb{C}^N$. Let B be a ball containing K such that $bB \cap \widehat{\Omega}$ is transversal. By hypothesis, we can choose a finite family of Zariski-open sets $V_1, \ldots, V_k$ which are uniformized by Stein manifolds and $\bigcap F_i$ is empty near bB, where $F_i = \Omega \setminus V_i$. By Kontinuitätssatz, F_i is a hypersurface of Ω. We also have $\overline{F}_i \subset F_i \cup K$. It is sufficient to prove that $F_i \cup K$ is a subvariety of $\widehat{\Omega}$. By slicing, one reduces the problem to the case of dimension 2.

Set $F = \bigcup F_i$. Observe that $\Gamma = F \cap bB$ is an analytic real curve. The classical Wermer theorem [19] says that $\mathrm{hull}(\Gamma) \setminus \Gamma$ is an analytic subset of pure dimension 1 of $\mathbb{C}^N \setminus \Gamma$ where $\mathrm{hull}(\Gamma)$ is the polynomial hull of Γ. By the uniqueness theorem, $\mathrm{hull}(\Gamma) \subset \widehat{\Omega}$. Since S is finite, $\mathrm{hull}(\Gamma \cup S) = \mathrm{hull}(\Gamma) \cup S$. Set $F' = (F \cup K) \cap \overline{B}$ and $F'' = \mathrm{hull}(\Gamma) \cup S$. We have $F' \subset F''$. Indeed, if $F' \not\subset F''$, using the maximum principle and some standard techniques, we construct a smooth family of discs which does not satisfy the Kontinuitätssatz.

Now, since $(F_i \cup K) \setminus S$ is contained in a hypersurface of $\widehat{\Omega} \setminus S$ and satisfies the Kontinuitätssatz, we can deduce that $(F_i \cup K) \setminus S$ is a hypersurface of $\widehat{\Omega} \setminus S$. By the Remmert–Stein theorem, any analytic set can be extended through a point, so $F_i \cup K$ is a hypersurface of $\widehat{\Omega}$. □

Proof of Theorem 3. Let X be a hyperconcave end such that the exhaustion function φ is overall strongly plurisubharmonic. Let $\widehat{X}$ be a manifold which compactifies X. Then $\widehat{X} \setminus X$ has a strongly pseudoconvex neighbourhood V. By [8, Satz 3, p. 338] there exists a maximal analytic set A of V and [8, Satz 5, p. 340] shows the existence of a normal Stein space V' with at worst isolated singularities, a discrete set $D \subset V'$ and a proper holomorphic map $\pi : V \to V'$, biholomorphic between $V \setminus A$ and $V' \setminus D$ and $\pi(A) = D$. That is, A can be blown down to the finite set D. Of course, $\mathrm{Sing}(V') \subset D$. The maximum principle for φ implies $A \subset \widehat{X} \setminus X$. Let $\psi : V' \to [-\infty, \infty)$ be given by $\psi = \varphi \circ \pi^{-1}$ on $V' \setminus D$ and $\psi = -\infty$ on $\pi(\widehat{X} \setminus X)$. Then ψ is a plurisubharmonic function on V' by [4] and $\pi(\widehat{X} \setminus X)$ is its pluripolar set. By Proposition 9, $\pi(\widehat{X} \setminus X)$ is a finite set. Therefore $\widehat{X} \setminus X$ consists of A and possibly a finite set D'. If X has a Stein cover, it follows from the Kontinuitätssatz that $\pi(\widehat{X} \setminus X) = \mathrm{Sing}(V')$. Therefore $D' = \varnothing$ and $D = \mathrm{Sing}(V')$. □

Proof of Corollary 4. From (i) and (ii) follows via the embedding theorem of Andreotti–Tomassini [2, Th. 2, p. 97], [15, Lemma 2.1] that X is biholomorphic to an open set of a projective manifold. We conclude by Theorem 3. □

As a consequence we get a stronger form of [18, Main Theorem] (also noted by Nadel [14] for manifolds of dimension greater than three):

Corollary 10. *Let X be a complete Kähler manifold of finite volume and bounded negative sectional curvature. If $\dim X \geqslant 2$, X is biholomorphic to a quasiprojective manifold which can be compactified to a Moishezon space by adding finitely many singular points.*

Indeed, the same argument as in [18] or [15, §3] shows, with the help of the Busemann function, that X is hyper 1-concave. Moreover, the negativity of the curvature implies that K_X is positive and the universal cover of X is Stein. We can thus apply Corollary 4.

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