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Differential Geometry

A canonical frame for nonholonomic rank two distributions of maximal class

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Abstract

In 1910 E. Cartan constructed the canonical frame and found the most symmetric case for maximally nonholonomic rank 2 distributions in $\mathbb{R}^5$. We solve the analogous problems for rank 2 distributions in $\mathbb{R}^n$ for arbitrary $n > 5$. Our method is a kind of symplectification of the problem and it is completely different from the Cartan method of equivalence. ***To cite this article: B. Doubrov, I. Zelenko, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Un repère canonique pour les distributions non holonômes de rang 2 de classe maximale. En 1910 E. Cartan a construit un repère canonique et a trouvé le cas le plus symétrique des distributions de rang 2 et non holonômes de manière maximale dans $\mathbb{R}^5$. Nous résolvons ici des problèmes analogues pour les distributions de rang 2 dans $\mathbb{R}^n$ avec $n > 5$ arbitraire. Notre méthode est une sorte de symplectification du problème et est complètement différente de la méthode par équivalence de Cartan. ***Pour citer cet article : B. Doubrov, I. Zelenko, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Une distribution vectorielle D de rang l sur une variétée M de dimension n ou une (l, n)-distribution (avec $l < n$) est un sous fibré du fibré tangent TM dont les fibres sont de dimension l. Le groupe des germes des difféomorphismes de M agit naturellement sur l'ensemble des germes des (l, n)-distributions et y définit une relation d'équivalence. La question est : *quand deux germes de distributions sont-ils équivalents* ? Les distributions sont naturellement associées aux systèmes de Pfaff et aux systèmes de contrôle linéaires dans les contrôles. L'invariant évident discret (mais très grossier dans la plupart des cas) d'une distribution D au point q est ce qu'on appelle *le vecteur de petite croissance* au point q : C'est la suite $\{\dim D^j(q)\}_{j\in\mathbb{N}}$ où D^j est la $j$ème puissance de D, c.-à-d., $D^j = D^{j-1} + [D, D^{j-1}]$, $D^1 = D$. Il est bien connu que génériquement les germes de distributions sont stables seulement dans les cas suivants : $l = 1$,

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$l = n - 1$ et $(l, n) = (2, 4)$. Dans tous les autres cas, des invariants fonctionnels apparaissent pour les distributions génériques. Dans le présent article nous nous restreignons aux distributions de rang 2. Le cas $n = 5$ (cas de la plus petite dimension pour laquelle des invariants fonctionnels apparaissent) a été traité par E. Cartan dans [3]. D'abord, pour toute $(2, 5)$-distribution dont le vecteur de petite croissance est $(2, 3, 5)$ il construit le corepère canonique dans une certaine variété de dimention 14 ce qui implique que le groupe de symétrie d'une telle distribution est au plus de dimention 14. Ensuite, il montre que toute $(2, 5)$-distribution qui admet un groupe de symétrie de dimension 14 est localement équivalente à la distribution associée à l'EDO sous-déterminée $z'(x) = (y''(x))^2$ et que son groupe de symétrie est isomorphe au groupe de Lie exceptionnel G_2 (c'est la première apparition naturelle de ce groupe). Après les travaux de Cartan, le problème était de construire un repère canonique et de trouver les cas les plus symétriques pour les $(2, n)$-distributions lorsque $n > 5$. Dans le présent article, nous résolvons ce problème. Nos constructions sont basées sur une approche variationelle dévelopée dans [2] et [6]. Grossièrement parlant, nous faisons une sorte de symplectification du problème en passant sur le fibré cotangent T^*M de la variété M.

Supposons que $\dim D^2(q) = 3$ et que $\dim D^3(q) > 3$ pour tout $q \in M$. Notons $(D^l)^\perp \subset T^*M$ l'annulateur de D^l : $(D^l)^\perp = \{(q, p) \in T^*M\colon\ p \cdot v = 0\ \forall v \in D^l(q)\}$. D'abord nous mettons en évidence la distribution caractéristique $\mathcal{C}$ de rang 1 sur la sous variété $(D^2)^\perp \setminus (D^3)^\perp$ de codimension 3 dans T^*M en prenant les noyaux de la restriction de forme symplectique standard de T^*M sur $(D^2)^\perp$. La distribution $\mathcal{C}$ définit un feuilletage de $(D^2)^\perp \setminus (D^3)^\perp$ dont les feuilles unidimensionelles sont appelées *les courbes caractéristiques* (ou extrémales anormales) de la distribution D. Ensuite, à tout point $q \in M$ nous associons un nombre naturel, une mesure de non holonômie, appelé *classe de D au point q*. Pour cela, notons $\mathcal{J}(\lambda) = \{v \in T_\lambda (D^2)^\perp\colon\ \pi_* v \in D(\pi(\lambda))\}$. Remarquons que $\dim \mathcal{J}(\lambda) = n - 1$. Définissons la suite de sous-espaces $\mathcal{J}^{(i)}(\lambda)$, $\lambda \in (D^2)^\perp \setminus (D^3)^\perp$, par la formule de récurrence : $\mathcal{J}^{(i)}(\lambda) = \mathcal{J}^{(i-1)}(\lambda) + \{[H, V](\lambda)\colon H \in \mathcal{C}, V \in \mathcal{J}^{(i-1)}\}$, $\mathcal{J}^{(0)} = \mathcal{J}$. Il suit de [6], Proposition 3.1 que, $\dim \mathcal{J}^{(1)}(\lambda) - \dim \mathcal{J}(\lambda) = 1$, dont on déduit facilement que $\dim \mathcal{J}^{(i)}(\lambda) - \dim \mathcal{J}^{(i-1)}(\lambda) \leqslant 1$ pour tout $i \in \mathbb{N}$. En outre, il est facile de montrer que $\dim \mathcal{J}^{(i)}(\lambda) \leqslant 2n - 4$ pour tout entier naturel i. Notons pour tout point $q \in M$ $(D^l)^\perp(q) = (D^l)^\perp \cap T_q^*M$ et définissons les deux fonctions à valeurs entières suivantes : $\nu(\lambda) = \min\{i \in \mathbb{N}\colon\ \mathcal{J}^{(i+1)}(\lambda) = \mathcal{J}^{(i)}(\lambda)\}$, et $m(q) = \max\{\nu(\lambda)\colon \lambda \in (D^2)^\perp(q) \setminus (D^3)^\perp(q)\}$. L'entier naturel $m(q)$ est appelé *la classe de la distribution D au point q*. Observons que $1 \leqslant m(q) \leqslant n - 3$. On peut montrer facilement que *les germes des $(2, n)$-distributions de classe maximale $n - 3$ sont génériques* (voir [6, Proposition 3.4]).

À partir de maintenant D sera une $(2, n)$-distribution de classe maximale constante $m = n - 3$. Soient $\mathcal{R} = \{\lambda \in (D^2)^\perp \setminus (D^3)^\perp\colon\ \nu(\lambda) = n - 3\}$ et $\mathcal{R}(q) = \mathcal{R} \cap T_q^*M$. Remarquons que $\mathcal{R}(q)$ est un ouvert non vide de $(D^2)^\perp(q)$ pour la topologie de Zariski (voir de nouveau [6], Proposition 3.4). Le point crucial et que *tout segment, inclu dans $\mathcal{R}$, d'une courbe caractéristique γ de D peut être muni d'une structure projective canonique* (voir [1,2], et [6]). Nous appelons structure projective sur une courbe l'ensemble de ses paramétrisations tel que le passage d'une paramétrisation à une autre se fasse par transformation de Möbius. Pour constuire la structure projective canonique d'une courbe γ on associe tout d'abord à γ une courbe particulière $\tilde{J}_\gamma$ de la grassmannian lagrangienne $L(W)$ de l'espace symplectic W de dimension $2m$, appelée courbe de Jacobi, et on construit ensuite la structure projective sur $\tilde{J}_\gamma$ (et par conséquent sur γ) en utilisant la notion de birapport généralisé de quatre points de $L(W)$. Nous sommes maintenant prêts à décrire les variétés sur lesquelles le repère canonique pour la $(2, n)$-distribution de classe maximale, $n > 5$, peut être construit. Étant donné un point $\lambda \in \mathcal{R}$ notons $\mathfrak{P}_\lambda$ l'ensemble de toutes les paramétrizations projectives $\varphi : \gamma \mapsto \mathbb{R}$ de la courbe caracteristique γ passant par λ telles que $\varphi(\lambda) = 0$. Notons $\Sigma = \{(\lambda, \varphi)\colon\ \lambda \in \mathcal{R}, \varphi \in \mathfrak{P}_\lambda\}$. En fait, Σ est un fibré principal de dimension $2n - 1$ sur $\mathcal{R}$ muni du groupe structurel de toutes les transformations de Möbius presérvant 0.

Théorème. *Pour toute $(2, n)$-distribution, $n > 5$, de classe maximale il existe deux repères canoniques sur la variété correpondante Σ de dimension $(2n - 1)$ obtenus l'un de l'autre par réflexion. Le groupe de symétrie d'une telle distribution est au plus de dimension $(2n - 1)$. Toute $(2, n)$-distribution de classe maximale dont le groupe de symétrie a dimension $(2n - 1)$ est localement équivalente à la distribution associée à l'EDO sous-déterminée $z'(x) = (y^{(n-3)}(x))^2$. L'algèbre des symétries infinitésimales de cette distribution est isomorphe à une somme semidirecte de $\mathfrak{gl}(2, \mathbb{R})$ et de l'algèbre de Heisenberg $\mathfrak{n}_{2n-5}$ de dimension $2n - 5$.*

La preuve consiste en la construction explicite des repères canoniques. Elle est basée sur le fait qu'on peut associer à la courbe de Jacobi d'une courbe caracteristique d'une distribution de rang 2, une courbe de drapeaux complets et qu'on peut normaliser, au signe près, les sous-espaces de dimension un de ces drapeaux en utilisant la structure symplectique.

1. Introduction

A rank l vector distribution D on an n-dimensional manifold M or an (l,n)-distribution (where $l<n$) is an subbundle of the tangent bundle TM with l-dimensional fibers. The group of germs of diffeomorphisms of M acts naturally on the set of germs of (l,n)-distributions and defines the equivalence relation there. The question is *when two germs of distributions are equivalent*? Distributions are naturally associated with Pfaffian systems and with control systems linear in the control. The obvious (but very rough in the most cases) discrete invariant of a distribution D at q is a so-called *the small growth vectors* at q. It is the tuple $\{\dim D^j(q)\}_{j\in\mathbb{N}}$, where D^j is the j-th power of the distribution D, i.e., $D^j=D^{j-1}+[D,D^{j-1}]$, $D^1=D$. It is well known that generic germs of distributions are stable only in the following three cases: $l=1$, $l=n-1$, and $(l,n)=(2,4)$. In all other cases generic (l,n)-distributions have functional invariants.

In the present Note we restrict ourselves to the case of rank 2 distributions. The model examples of such distributions come from so-called underdetermined ODE's of the type $z^{(r)}(x)=F(x,y(x),\dots,y^{(s)}(x),z(x),\dots,z^{(r-1)}(x))$, $r+s=n-2$, for two functions $y(x)$ and $z(x)$. Setting $p_i=y^{(i)}$, $0\leqslant i\leqslant s$, and $q_j=z^{(j)}$, $0\leqslant j\leqslant r-1$, with each such equation one can associate a rank 2 distribution in $\mathbb{R}^n$ with coordinates $(x,p_0,\dots,p_s,q_0,\dots,q_{r-1})$ given by the intersection of the annihilators of the following $n-2$ one-forms: $\mathrm{d}p_i-p_{i+1}\,\mathrm{d}x$, $0\leqslant i\leqslant s-1$, $\mathrm{d}q_j-q_{j+1}\,\mathrm{d}x$, $0\leqslant j\leqslant r-2$, and $\mathrm{d}q_{r-1}-F(x,p_0,\dots,p_s,q_0,\dots,q_{r-1})\,\mathrm{d}x$.

For $n\in\{3,4\}$ all generic germs of rank 2 distributions are equivalent to the distribution, associated with the underdetermined ODE $z'(x)=y(x)$ (Darboux and Engel models respectively). The case $n=5$ was treated by E. Cartan in [3] with his reduction-prolongation procedure. First, for any $(2,5)$-distribution with small growth vector $(2,3,5)$ he constructed the canonical coframe in some 14-dimensional manifold, which implied that the group of symmetries of such distributions is at most 14-dimensional. Second, he showed that any $(2,5)$-distribution with 14-dimensional group of symmetries is locally equivalent to the distribution, associated with the underdetermined ODE $z'(x)=(y''(x))^2$, and its group of symmetries is isomorphic to the real split form of the exceptional Lie group G_2 (it was the first natural appearance of this group).

After the work of Cartan the open question was *to construct the canonical frame and to find the most symmetric cases for $(2,n)$-distributions with $n>5$*. The Cartan equivalence method was systematized and generalized by N. Tanaka and T. Morimoto (see [5,4]). Their theory is heavily based on the notion of so-called *symbol algebra* of the distribution at a point, which is a special *graded nilpotent Lie algebra*, naturally associated with the distribution at a point. The symbol algebras have to be isomorphic at different points (i.e. the symbol has to be constant) and all constructions strongly depend on the type of symbol. Note that already in the case of $(2,6)$-distributions with maximal possible small growth vector $(2,3,5,6)$ three different symbol algebras are possible, while for $n=9$ the set of all possible symbol algebras depends on continuous parameters, which, in particular, implies that generic distributions do not have a constant symbol. In the present paper we give an answer to the question stated above for rank 2 distributions from some generic class. Our constructions are based on a completely different, variational approach, developed in [2] and [6]. Roughly speaking, we make a kind of symplectification of the problem by lifting the distribution to the cotangent bundle T^*M of the manifold M.

2. Characteristic curves, the class, and the canonical projective structure

Assume that $\dim D^2(q)=3$ and $\dim D^3(q)>3$ for any $q\in M$. Denote by $(D^l)^\perp\subset T^*M$ the annihilator of the l-th power D^l, namely $(D^l)^\perp=\{(q,p)\in T^*M\colon\ p\cdot v=0\ \forall v\in D^l(q)\}$. First, we distinguish a characteristic 1-foliation on the codimension 3 submanifold $(D^2)^\perp\backslash(D^3)^\perp$ of T^*M. For this let $\pi: T^*M\mapsto M$ be the canonical projection. For any $\lambda\in T^*M$, $\lambda=(p,q)$, $q\in M$, $p\in T^*_qM$, let $\mathfrak{s}(\lambda)(\cdot)=p(\pi_*\cdot)$ be the canonical Liouville form and $\sigma=d\mathfrak{s}$ be the standard symplectic structure on T^*M. Since the submanifold $(D^2)^\perp$ has odd codimension in T^*M, the kernels of the restriction $\sigma|_{(D^2)^\perp}$ of σ on $(D^2)^\perp$ are not trivial. Moreover for the points of $(D^2)^\perp\backslash(D^3)^\perp$ these kernels are one-dimensional. They form the *characteristic line distribution* in $(D^2)^\perp\backslash(D^3)^\perp$, which will be denoted by $\mathcal{C}$. The line distribution $\mathcal{C}$ defines a *characteristic 1-foliation* of $(D^2)^\perp\backslash(D^3)^\perp$. The leaves of this foliation are called the *characteristic curves* (or *the abnormal extremals*) of the distribution D (the second term comes from Optimal Control Theory).

Second, to any point $q\in M$ we assign a natural number, a measure of nonholonomy of D. For this let $\mathcal{J}(\lambda)=\{v\in T_\lambda(D^2)^\perp\colon\ \pi_*v\in D(\pi(\lambda))\}$. Note that $\dim\mathcal{J}(\lambda)=n-1$. Define a sequence of subspaces $\mathcal{J}^{(i)}(\lambda)$, $\lambda\in(D^2)^\perp\backslash(D^3)^\perp$, by the following recursive formulas: $\mathcal{J}^{(i)}(\lambda) = \mathcal{J}^{(i-1)}(\lambda) + \{[H, V](\lambda)\colon\ H \in \mathcal{C}, V \in \mathcal{J}^{(i-1)}\}$, $\mathcal{J}^{(0)} = \mathcal{J}$. By [6, Proposition 3.1], we have $\dim \mathcal{J}^{(1)}(\lambda) - \dim \mathcal{J}(\lambda) = 1$, which implies easily that $\dim \mathcal{J}^{(i)}(\lambda) - \dim \mathcal{J}^{(i-1)}(\lambda) \leqslant 1$ for all $i \in \mathbb{N}$. Besides, it is not difficult to show that $\mathcal{J}^{(i)} \subset \{\mathfrak{s}|_{(D^2)^\perp} = 0\}$ for all natural i, which yields that $\dim \mathcal{J}^{(i)}(\lambda) \leqslant 2n - 4$. Further for any point $q \in M$ denote by $(D^l)^\perp(q) = (D^l)^\perp \cap T_q^* M$. Let us define the following two integer-valued functions: $\nu(\lambda) = \min\{i \in \mathbb{N}\colon\ \mathcal{J}^{(i+1)}(\lambda) = \mathcal{J}^{(i)}(\lambda)\}$, and $m(q) = \max\{\nu(\lambda)\colon\ \lambda \in (D^2)^\perp(q) \backslash (D^3)^\perp(q)\}$. The number $m(q)$ is called *the class of distribution D at the point q*. By above, $1 \leqslant m(q) \leqslant n - 3$. It is easy to show that *germs of* $(2, n)$*-distributions of the maximal class* $n - 3$ *are generic* (see [6, Proposition 3.4]).

From now on we assume that D is a $(2, n)$-distribution of maximal constant class $m = n - 3$. Let $\mathcal{R} = \{\lambda \in (D^2)^\perp \backslash (D^3)^\perp\colon\ \nu(\lambda) = n - 3\}$, and $\mathcal{R}(q) = \mathcal{R} \cap T_q^* M$. The set $\mathcal{R}(q)$ is a nonempty open set in Zariski topology on the linear space $(D^2)^\perp(q)$ (see again [6, Proposition 3.4]).

The crucial observation is that *any segment of a characteristic curve* γ *of D, belonging to* $\mathcal{R}$*, can be endowed with a canonical projective structure* (see also [1,2], and [6]). By a projective structure on a curve we mean a set of parameterizations such that the transition function from one such parameterization to another is a Möbius transformation. To construct this canonical projective structure on γ we first associate with γ a special curve in a Grassmannian $G_m(W)$ of m-dimensional subspaces of a $2m$-dimensional linear space W. This curve is called *the Jacobi curve* and is constructed in the following way. Let O_γ be a neighborhood of γ in $(D^2)^\perp$ such that the factor $N = O_\gamma/($*the characteristic one-foliation*$)$ is a well-defined smooth manifold. Its dimension is equal to $2(n - 2)$. Let $\phi\colon O_\gamma \to N$ be the canonical projection on the factor.

Define the mapping $J_\gamma\colon \gamma \mapsto G_{n-2}(T_\gamma N)$ by $J_\gamma(\lambda) = \phi_*(\mathcal{J}(\lambda))$ for all $\lambda \in \gamma$, where $\mathcal{J}(\lambda)$ is as above. Actually, the symplectic form σ of T^*M induces naturally the symplectic form $\bar{\sigma}$ on $T_\gamma N$ and $J_\gamma(\lambda)$ for $\lambda \in \gamma$ are Lagrangian subspace of $T_\gamma N$. Besides, if e is the Euler field (i.e., the infinitesimal generator of homotheties on the fibers of T^*M), then the vector $\bar{e} = \phi_* e(\lambda)$ is the same for any $\lambda \in \gamma$ and lies in $J_\gamma(\lambda)$. Therefore the curve $\lambda \mapsto \tilde{J}_\gamma(\lambda) = J_\gamma(\lambda)/\{\mathbb{R}\bar{e}\}$, $\lambda \in \gamma$, is a curve in $G_m(W)$, where $W = \{v \in T_\gamma N\colon\ \bar{\sigma}(v, \bar{e}) = 0\}/\{\mathbb{R}\bar{e}\}$. The curve $\tilde{J}_\gamma$ is called *the Jacobi curve of* γ.

Second, we construct the canonical projective structure on $\tilde{J}_\gamma$ (and therefore on γ itself), using the notion of the generalized cross-ratio of 4 points in $G_m(W)$. Namely, let $\{\Lambda_i\}_{i=1}^4$ be any 4 points of $G_m(W)$. For simplicity suppose that $\Lambda_i \cap \Lambda_j = 0$ for $i \neq j$. Assume that in some coordinates $W \cong \mathbb{R}^m \times \mathbb{R}^m$ and $\Lambda_i = \{(x, S_i x)\colon\ x \in \mathbb{R}^m\}$ for some $m \times m$-matrix S_i. Then the conjugacy class of the following matrix $(S_1 - S_4)^{-1}(S_4 - S_3)(S_3 - S_2)^{-1}(S_2 - S_1)$ does not depend on the choice of the coordinates in W. This conjugacy class is called *the cross-ratio of the tuple* $\{\Lambda_i\}_{i=1}^4$ and it is denoted by $[\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4]$. Take now some parameterization $\varphi\colon \gamma \mapsto \mathbb{R}$ of γ and let $\Lambda_\varphi(t) = \tilde{J}_\gamma(\varphi^{-1}(t))$. Assume that in some coordinates on W we have $\Lambda_\varphi(t) = \{(x, S_t x)\colon\ x \in \mathbb{R}^m\}$. The following fact follows from [6, Proposition 2.1]. For all parameters t_1 the functions $t \to \det(S_t - S_{t_1})$ have zero of the same order m^2 at $t = t_1$. Consider the following function $\mathcal{G}_{\Lambda_\varphi}(t_1, t_2, t_3, t_4) = \ln(\det[\Lambda_\varphi(t_1), \Lambda_\varphi(t_2), \Lambda_\varphi(t_3), \Lambda_\varphi(t_4)]([t_1, t_1, t_2, t_3])^{-m^2})$, where $[t_1, t_2, t_3, t_4] = \frac{(t_2-t_1)(t_4-t_3)}{(t_3-t_2)(t_1-t_4)}$ is the usual cross-ratio of 4 numbers $\{t_i\}_{i=1}^4$. Then, by above, it is not hard to see that $\mathcal{G}_{\Lambda_\varphi}(t_1, t_2, t_3, t_4)$ is smooth at diagonal points (t, t, t, t) and the Taylor expansions of it at these points have the form $\mathcal{G}_{\Lambda_\varphi}(t_0, t_1, t_2, t_3) = \rho_{\Lambda_\varphi}(t)(\xi_1 - \xi_3)(\xi_2 - \xi_4) + \cdots$, where $\xi_i = t_i - t$. Analyzing how the function $\rho_{\Lambda_\varphi}(t)$ transforms when one takes another parameterization of γ, it is not hard to show (see [1] or [2]) that *the set of all parameterizations* φ *of* γ *such that* $\rho_{\Lambda_\varphi}(t) \equiv 0$ *defines the canonical projective structure on* γ.

3. The main theorem

Now we are ready to describe the manifold, on which the canonical frame for $(2, n)$-distribution of maximal class, $n > 5$, can be constructed. Given $\lambda \in \mathcal{R}$ denote by $\mathfrak{P}_\lambda$ the set of all projective parameterizations $\varphi\colon \gamma \mapsto \mathbb{R}$ on the characteristic curve γ, passing through λ, such that $\varphi(\lambda) = 0$. Denote $\Sigma = \{(\lambda, \varphi)\colon\ \lambda \in \mathcal{R}, \varphi \in \mathfrak{P}_\lambda\}$. Actually, Σ is a principal bundle over $\mathcal{R}$ with the structural group of all Möbius transformations preserving 0, and $\dim \Sigma = 2n - 1$.

Theorem. *For any* $(2, n)$*-distribution,* $n > 5$*, of maximal class there exist two canonical frames on the corresponding* $(2n - 1)$*-dimensional manifold* Σ*, obtained one from another by a reflection. The group of symmetries of such distribution is at most* $(2n - 1)$*-dimensional. Any* $(2, n)$*-distribution of maximal class with* $(2n - 1)$*-dimensional group of symmetries is locally equivalent to the distribution, associated with the underdetermined ODE* $z'(x) = (y^{(n-3)}(x))^2$.

The algebra of infinitesimal symmetries of this distribution is isomorphic to a semidirect sum of $\mathfrak{gl}(2,\mathbb{R})$ *and* $(2n-5)$*-dimensional Heisenberg algebra* $\mathfrak{n}_{2n-5}$.

Sketch of the proof. Define the following two fiber-preserving flows on Σ: $F_{1,s}(\lambda,\varphi)=(\lambda,\mathrm{e}^{2s}\varphi)$ and $F_{2,s}(\lambda,\varphi)=(\lambda,\frac{\varphi}{-s\varphi+1})$, where $\lambda\in\mathcal{R}$ and $\varphi\in\mathfrak{P}_\lambda$. Further, let δ_s be the flow of homotheties on the fibers of T^*M: $\delta_s(p,q)=(\mathrm{e}^s p,q)$, where $q\in M$, $p\in T_q^*M$ (actually the Euler field e generates this flow). The following flow $F_{0,s}(\lambda,\varphi)=(\delta_{2s}(\lambda),\varphi\circ\delta_{2s}^{-1})$ is well-defined on Σ (here we use the fact that δ_s preserves the characteristic 1-foliation). For any $0\leqslant i\leqslant 2$ let g_i be the vector field on Σ, generating the flow $F_{i,s}$. Besides, the characteristic 1-foliation on $(D^2)^\perp$ can be lifted to the *parameterized* 1-foliation on Σ, which gives one more canonical vector field on Σ. Indeed, let $u=(\lambda,\varphi)\in\Sigma$ and γ be the characteristic curve, passing through λ (so, φ maps γ to $\mathbb{R}$). Then the mapping $\Gamma_u(t)=(\varphi^{-1}(t),\varphi(\cdot)-t)$ defines the parameterized curve on Σ, the lift of γ to Σ, and $\Gamma_u(0)=u$. The additional canonical vector field h on Σ is defined by $h(u)=\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_u(t)|_{t=0}$. It can be shown easily that $[g_1,g_2]=2g_2$, $[g_1,h]=-2h$, $[g_2,h]=g_1$, and g_0 commutes with all constructed fields. Therefore $\operatorname{span}_\mathbb{R}\{g_0,g_1,g_2,h\}$ is isomorphic to $\mathfrak{gl}(2,\mathbb{R})$.

Now we will construct one more canonical, up to the sign, vector field on Σ. For this let $\mathcal{J}_{(i)}(\lambda)=\{v\in T_\lambda((D^2)^\perp)\colon \sigma(v,w)=0\,\forall w\in\mathcal{J}^{(i)}\}$ and $V_i(\lambda)=\{\lambda\in\mathcal{J}_{(i)}\colon \pi_*(v)=0\}$. Since $\mathcal{J}^{(i)}\subseteq\mathcal{J}^{(i+1)}$, we have $\mathcal{J}_{(i+1)}\subseteq\mathcal{J}_{(i)}$. If $\lambda\in\mathcal{R}$, then $\dim\mathcal{J}^{(i)}=n-1+i$, which implies that $\dim\mathcal{J}_{(i)}=n-1-i$. Besides, it is easy to show that $\mathcal{J}_{(i)}=V_i\oplus\mathcal{C}$. Therefore $\dim V_i=n-2-i$. In particular, $\dim V_{n-4}=2$. Also the Euler field $e\in V_{(n-4)}$. Fix a point $\lambda\in\mathcal{R}$. Let γ be the characteristic curve, passing through λ, and let φ be a parameterization on γ such that $\varphi(\lambda)=0$. As before, let $m=n-3$. Then there exists a vector $\varepsilon_\varphi(\lambda)\in V_{n-4}(\lambda)$ such that if E and H are two vector fields, satisfying $E\in V_{n-4}$, $H\in\mathcal{C}$, $E(\lambda)=\varepsilon_\varphi(\lambda)$, and $H(\lambda)=\frac{\mathrm{d}}{\mathrm{d}t}\varphi^{-1}(t)|_{t=0}$, then $|\sigma((\operatorname{ad}H)^m E(\lambda),(\operatorname{ad}H)^{m-1}E(\lambda))|=1$. Such vector is defined up to the transformations $\varepsilon_\varphi(\lambda)\to\pm\varepsilon_\varphi(\lambda)+\mu e(\lambda)$. Denote by $\Pi:\Sigma\mapsto\mathcal{R}$ the canonical projection. Let ε_1 be a vector field on Σ such that for all $u=(\lambda,\varphi)\in\Sigma$ one has $\Pi_*\varepsilon_1(u)=\pm\varepsilon_\varphi(\lambda) \bmod \{\mathbb{R}e(\lambda)\}$. Such fields ε_1 are defined modulo $\operatorname{span}\{g_0,g_1,g_2\}$ and the sign. How to choose among them the canonical field, up to the sign? Fix one of such fields ε_1. Denote by $\varepsilon_i=(\operatorname{ad}h)^{i-1}\varepsilon_1$ for $2\leqslant i\leqslant 2m$ and $\eta=[\varepsilon_1,\varepsilon_{2m}]$. First the tuple $(h,\{g_i\}_{i=0}^2,\{\varepsilon_i\}_{i=1}^{2m},\eta)$ is a frame on Σ. Let $\mathcal{L}_j=\operatorname{span}\{h,\{g_i\}_{i=0}^2,\{\varepsilon_i\}_{i=1}^j\}$ for $0\leqslant j\leqslant 2m$. Then one can show that $[\varepsilon_1,\varepsilon_2]=\kappa_1\varepsilon_2 \bmod \mathcal{L}_1$ and, in the case $n>5$, $[\varepsilon_1,\varepsilon_4]=\kappa_2\varepsilon_3+\kappa_3\varepsilon_4 \bmod \mathcal{L}_2$. It turns out that among all fields ε_1 there exists the unique, up to the sign, field $\tilde\varepsilon_1$ such that the functions κ_i, $1\leqslant i\leqslant 3$, are identically zero. Then two frames $(h,\{g_i\}_{i=0}^2,\{\tilde\varepsilon_i\}_{i=1}^{2m},\eta)$ and $(h,\{g_i\}_{i=0}^2,\{-\tilde\varepsilon_i\}_{i=1}^{2m},\eta)$ are canonically defined. This immediately implies that the groups of symmetries is at most $(2n-1)$-dimensional.

If a $(2,n)$-distribution of maximal class has a $(2n-1)$-dimensional group of symmetries, then all structural functions of its canonical frames have to be constant. It can be shown that the only nonzero commutative relations of each of these frames in addition to the mentioned above are $[\tilde\varepsilon_i,\tilde\varepsilon_{2m-i+1}]=(-1)^{i+1}\eta$, $[g_1,\tilde\varepsilon_i]=(2m-2i+1)\tilde\varepsilon_i$, $[g_2,\tilde\varepsilon_i]=(i-1)(2m+1-i)\tilde\varepsilon_{i-1}$, $[g_0,\tilde\varepsilon_i]=-\tilde\varepsilon_i$, and $[g_0,\eta]=-2\eta$, which implies the uniqueness of such distribution, up to the equivalence. Besides, from these relations it follows that the algebra of infinitesimal symmetries of such distribution is isomorphic to the semi-direct sum of $\mathfrak{gl}(2,\mathbb{R})$ ($\sim\operatorname{span}_\mathbb{R}\{g_0,g_1,g_2,h\}$) and the Heisenberg group $\mathfrak{n}_{2m+1}$ ($\sim\operatorname{span}_\mathbb{R}\{\tilde\varepsilon_1,\dots,\tilde\varepsilon_{2m},\eta\}$). Finally, it is easy to show that for $(2,n)$-distribution, associated with $z'(x)=(y^{(n-3)}(x))^2$, the canonical frames satisfy the previous commutative relations. □

4. Open question

What happens with rank 2 distribution of non-maximal class? From Remark 3.4 of [6] it follows that a rank 2 distribution D has minimal class 1 at a point q iff $\dim D^3(q)=4$. On the other hand, if D satisfies $\dim D^3(q)=4$ on some open set, then either it is the Goursat distribution and has an infinite-dimensional symmetry algebra or by the factorization of the ambient manifold by the characteristics of D^2 (or series of such factorizations) one can get a distribution $\widetilde{D}$, satisfying $\dim\widetilde{D}^3=5$. In other words, the case of non-Goursat distributions of constant class 1 can be reduced to the case of distributions of class greater than 1. Do there exist completely nonholonomic rank 2 distributions of constant class $2\leqslant m\leqslant n-4$? We know only that the answer is negative for $m=2$ ($n>5$), which means that any such example, if it exists, should live on at least 7-dimensional manifold.

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