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Probability Theory

# Invariance principle for a class of non stationary processes with long memory ☆

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**Abstract**

We prove a functional central limit theorem for the partial sums of a class of time varying processes with long memory. ***To cite this article: A. Philippe et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***


**Résumé**

**Principe d'invariance pour des processus non stationnaires à longue mémoire.** Nous étudions une famille de processus non stationnaires à longue mémoire. Nous prouvons un théorème limite fonctionnel pour le processus des sommes partielles. ***Pour citer cet article : A. Philippe et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***


## Version française abrégée

### *Introduction*

Dans [3], Philippe et al. ont introduit une nouvelle classe d'opérateurs linéaires définis à partir d'une suite $\mathbf{d} = (d_t)_{t\in\mathbb{Z}}$ par $A(\mathbf{d})x_t = \sum_{s\leqslant t} a(s,t)x_s$, et $B(\mathbf{d})x_t = \sum_{s\leqslant t} b(s,t)x_s$, où

$$\forall s<t: \quad a(s,t) := \prod_{s\leqslant u<t} \frac{t-u-1-d_u}{t-u}, \quad b(s,t) := -d_{t-1} \prod_{s\leqslant u<t-1} \frac{u-s+1-d_u}{u-s+2},$$

et $a(t,t) = b(t,t) = 1$. Le domaine des opérateurs est étudié dans la section 2. Ils sont liés par la relation $B(-\mathbf{d})A(\mathbf{d}) = A(-\mathbf{d})B(\mathbf{d}) = I$. De plus, lorsque $\mathbf{d}$ est constante, $A(\mathbf{d})$ et $B(\mathbf{d})$ coïncident avec le filtre fraction-

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naire $(I-L)^d$ (où $L$ est l'opérateur retard). Soit $(\varepsilon_t)$ une suite de variables i.i.d. centrées et de variance 1. L'existence dans $L^2$ de solutions aux équations $A(\mathbf{d})X_t=\varepsilon_t$ et $B(\mathbf{d})Y_t=\varepsilon_t$ est prouvée dans [3], où l'on montre aussi que lorsque $\mathbf{d}$ admet des limites $d_\pm \in (0,1/2)$ en $\pm\infty$, les sommes partielles de ces processus non stationnaires convergent vers un processus auto-similaire gaussien dont les accroissements ne sont pas stationnaires, contrairement à la situation $\mathbf{d}$ constante où la limite est le mouvement brownien fractionnaire.

On s'intéresse ici aux suites $\mathbf{d}$ bornées *moyennable* c'est à dire telles que

$$n^{-1}\sum_{k=s}^{s+n} d_k \overset{n\to+\infty}{\longrightarrow} \bar{d} \quad \text{uniformément en } s\in\mathbb{Z}.$$

Soit $\mathcal{M}$ une sous classe des suites moyennables bornées, fermée pour les opérations algébriques, par translation et pour la convergence uniforme (par exemple, la classe des suites presque périodiques convient). Nous montrons que pour $\mathbf{d}\in\mathcal{M}$, le processus $(X_t)$ (resp. $(Y_t)$) solution de $A(\mathbf{d})X_t=\varepsilon_t$ (resp. $B(\mathbf{d})Y_t=\varepsilon_t$) possède des propriétés asymptotiques similaires à celle d'un processus fractionnaire de paramètre $\bar{d}$.

*Préliminaires*

Nous commençons par étudier le comportement des coefficients $a^-(s,t)$ (resp. $b^-(s,t)$) de l'opérateur $A(-\mathbf{d})$ (resp. $B(-\mathbf{d})$).

**Lemme 0.1.** *Soit*

$$\bar{d}_+ := \limsup_{t-s\to\infty} \frac{1}{t-s}\sum_{s<u<t} d_u.$$

*Alors, pour tout $D>\bar{d}_+$, $D\notin\mathbb{Z}_-$ il existe une constante $C<\infty$ telle que $\forall s\leqslant t$*

$$\left|a^-(s,t)\right| + \left|b^-(s,t)\right| \leqslant C\left|\psi_{t-s}(-D)\right|,$$

*où les coefficients $\psi_j(-D)$ sont définis en* (2).

Si l'on suppose que la valeur moyenne de $\mathbf{d}\in\mathcal{M}$ appartient à $(0,1/2)$, alors une conséquence immédiate de ce résultat est l'existence au sens de la convergence $L^2$ des processus

$$X_t = A(\mathbf{d})^{-1}\varepsilon_t = B(-\mathbf{d})\varepsilon_t \quad \text{et} \quad Y_t = B(\mathbf{d})^{-1}\varepsilon_t = A(-\mathbf{d})\varepsilon_t,$$

où $(\varepsilon_t,\ t\in\mathbb{Z})$ est une suite de variables aléatoires i.i.d., centrées et de variance 1.

Le résultat suivant donne un contrôle des coefficients $a^-(s,t)$ et $b^-(s,t)$ lorsque $t-s$ devient grand.

**Lemme 0.2.** *On suppose que la suite $\mathbf{d}\in\mathcal{M}$ vérifie la condition* (6) *avec $\bar{d}\in(0,1/2)$ et que pour tout $t\in\mathbb{Z}$, $d_t\notin\mathbb{Z}_-$. On a*

$$a^-(s,t) = q_A(t)\psi_{t-s}(-\bar{d})\theta_{s,t}, \qquad b^-(s,t) = q_B(s)\frac{d_{t-1}}{\bar{d}}\psi_{t-s}(-\bar{d})\vartheta_{s,t},$$

*où*

$$|\theta_{s,t}-1| + |\vartheta_{s,t}-1| \leqslant C|s-t|^{-\delta} (s<t).$$

*Les suites $q_A$ et $q_B$ sont définies en* (7). *Elles appartiennent à la classe $\mathcal{M}$.*

*Résultat principal*

Nous allons maintenant prouver un principe d'invariance pour les processus non stationnaires $(X_t)$ et $(Y_t)$ définis en (4), (5).

**Théorème 0.3.** *Soit* **d** *une suite vérifiant les hypothèses du Lemme* 0.2*. Les sommes partielles renormalisées des processus* $(X_t)$ *et* $(Y_t)$ *convergent faiblement dans l'espace de Skorohod* $D[0,1]$ :

$$N^{-\bar{d}-(1/2)}\sum_{t=1}^{[N\tau]} Y_t \xrightarrow{D[0,1]} c_A B_H(\tau), \quad et \quad N^{-\bar{d}-(1/2)}\sum_{t=1}^{[N\tau]} X_t \xrightarrow{D[0,1]} c_B B_H(\tau)$$

*où* $B_H$ *est le mouvement Brownien fractionnaire de paramètre* $H = \bar{d} + 1/2$*, et* $c_A$*,* $c_B$ *sont les constantes positives définies en* (12)*.*

Ce résultat de convergence des sommes partielles est très classique dans le sens où l'on retrouve le même processus limite et la même normalisation que pour les processus fractionnaires $Z_t = (I-L)^{-\bar{d}}\varepsilon_t$.

## 1. Introduction

Given a real sequence $\mathbf{d} = (d_t,\ t \in \mathbb{Z})$, in Philippe et al. [3], we introduce the time varying fractionally integrated linear operators $A(\mathbf{d})x_t := \sum_{s\leqslant t} a(s,t)x_s$, and $B(\mathbf{d})x_t := \sum_{s\leqslant t} b(s,t)x_s$, where for $s < t$

$$a(s,t) := \prod_{s\leqslant u<t} \frac{t-u-1-d_u}{t-u}, \qquad b(s,t) := -d_{t-1}\prod_{s\leqslant u<t-1} \frac{u-s+1-d_u}{u-s+2}, \tag{1}$$

and $a(t,t) = b(t,t) := 1$. The domain of operators $A(\mathbf{d})$ and $B(\mathbf{d})$ is studied in Section 2. If the sequence **d** is constant ($d_t \equiv d$), the operators $A(\mathbf{d})$ and $B(\mathbf{d})$ coincide with the usual fractional differentiation operator $(I-L)^d$,

$$(I-L)^d x_t = \sum_{j=0}^{\infty} \psi_j(d)x_{t-j} \quad \text{where } \forall j>0,\ \psi_j(d) = \frac{\Gamma(-d+j)}{j!\,\Gamma(-d)} \text{ and } \psi_0(d) = 1, \tag{2}$$

where $Lx_t = x_{t-1}$ is the backshift operator (see [2]). Moreover, $B(-\mathbf{d})A(\mathbf{d}) = A(-\mathbf{d})B(\mathbf{d}) = I$ and therefore $A(\mathbf{d})$ and $B(\mathbf{d})$ are invertible under general conditions on the sequence **d**.

In [3], we studied the long memory properties of $(X_t)$ and $(Y_t)$ defined respectively as solutions of equations $A(\mathbf{d})X_t = \varepsilon_t$, and $B(\mathbf{d})Y_t = \varepsilon_t$, where $(\varepsilon_t)$ is a standard i.i.d. sequence, when the sequence $\mathbf{d} = (d_t,\ t \in \mathbb{Z})$ has limits $\lim_{t\to\pm\infty} d_t =: d_\pm \in (0, 1/2)$. We show in [3] that the partial sums of $(X_t)$ and $(Y_t)$ converge to two different Gaussian self-similar processes depending on the asymptotic parameters $d_\pm$ only and having asymptotically stationary or asymptotically vanishing increments.

In the present paper we consider the case of bounded sequences **d** such that the following limit exists

$$\bar{d} = \lim_{n\to\infty} n^{-1}\sum_{k=s}^{s+n} d_k \quad \text{uniformly in } s \in \mathbb{Z}. \tag{3}$$

We say that a sequence **d** satisfying (3) is *averageable* and call $\bar{d}$ its *mean value*. Let $\mathcal{M}$ be a subclass of bounded averageable sequences such that $\mathcal{M}$ is closed under algebraic operations, shifts and uniform limits. For example, $\mathcal{M}$ can be the class of almost periodic sequences, or the class of all bounded sequences **d** such that $\lim_{t\to\infty} d_t = \lim_{t\to-\infty} d_t \in \mathbb{R}$.

As shown below, if $\mathbf{d} \in \mathcal{M}$ and if $\bar{d} \in (0, 1/2)$, then an 'averaging of nonstationary long memory' of $(X_t)$ and $(Y_t)$ occurs and large sample behavior of these processes is similar to the behavior of a stationary FARIMA$(0, \bar{d}, 0)$ process.

Everywhere below, we assume that $\mathbf{d} = (d_t,\ t \in \mathbb{Z})$ is bounded, $d_t \notin \mathbb{Z}_-$ for all $t \in \mathbb{Z}$, and that $(\varepsilon_t, t \in \mathbb{Z})$ is an i.i.d. sequence, with zero mean and unit variance.

## 2. Some preliminaries

Let $a^-(s,t)$, $b^-(s,t)$ be the coefficients defined in (1) with $\mathbf{d} = (d_t)$ replaced by $-\mathbf{d} = (-d_t)$.

**Lemma 2.1.** *Let*

$$\bar{d}_+ := \limsup_{t-s\to\infty} \frac{1}{t-s} \sum_{s<u<t} d_u.$$

*Then for any $D > \bar{d}_+$, $D \notin \mathbb{Z}_-$ there exists a constant $C < \infty$ such that for all $s \leqslant t$*

$$\left|a^-(s,t)\right| + \left|b^-(s,t)\right| \leqslant C\left|\psi_{t-s}(-D)\right|,$$

*where the fractional coefficients $(\psi_j(-D))$ are defined in* (2).

**Proof.** See [3]. □

Lemma 2.1 ensures that if $\mathbf{d}$ is averageable with mean value $\bar{d} < 1/2$, then the processes

$$X_t = A(\mathbf{d})^{-1}\varepsilon_t = B(-\mathbf{d})\varepsilon_t = \sum_{s\leqslant t} b^-(s,t)\varepsilon_s, \tag{4}$$

$$Y_t = B(\mathbf{d})^{-1}\varepsilon_t = A(-\mathbf{d})\varepsilon_t = \sum_{s\leqslant t} a^-(s,t)\varepsilon_s \tag{5}$$

are well-defined, in the sense of $L^2$ convergence of the series.

**Lemma 2.2.** *Let $\mathbf{d} \in \mathcal{M}$ with mean value $\bar{d} \in (0, 1/2)$. Assume that there exist $C$ and $0 < \delta < 1$ such that for all $s < t$*

$$\left|(t-s)^{-1}\sum_{u=s}^{t} d_u - \bar{d}\right| < C(t-s)^{-\delta}. \tag{6}$$

*Then the infinite products*

$$q_A(t) := \prod_{s<t}\left(1 + \frac{d_s - \bar{d}}{\bar{d} + t - s - 1}\right) \quad \text{and} \quad q_B(t) := \prod_{s\geqslant t}\left(1 + \frac{d_s - \bar{d}}{\bar{d} + s - t - 1}\right) \tag{7}$$

*converge uniformly in $t \in \mathbb{Z}$. Moreover, both sequences $(q_A(t),\ t \in \mathbb{Z})$ and $(q_B(t),\ t \in \mathbb{Z})$ belong to $\mathcal{M}$.*

**Proof.** To prove the uniform convergence of $q_A(t)$, it is enough to show that, as $n, m \to \infty$

$$\sup_{t\in\mathbb{Z}}\left|q_A^n(t) - q_A^m(t)\right| \to 0, \quad \text{where } q_A^n(t) := \prod_{p=1}^{n}\left(1 + \frac{d_{t-p} - \bar{d}}{\bar{d} + p - 1}\right) =: \prod_{p=1}^{n}\left(1 + \beta_p(t)\right). \tag{8}$$

The sequence $(d_t,\ t \in \mathbb{Z})$ is bounded, thus there exists $n_0$ such that $|\beta_p(t)| < 1/2$ for all $p \geqslant n_0$, $t \in \mathbb{Z}$. Since, the uniform convergence of $q_n^A(t)$ is equivalent to the uniform convergence of $q_n^A(t)/q_{n_0}^A(t)$. Therefore, we can suppose that $n_0 = 1$. Then $q_n^A(t) > 0$, for all $n \geqslant 1$ and

$$\left|q_A^n(t) - q_A^m(t)\right| \leqslant q_A^n(t)\left|\prod_{p=n+1}^{m}\left(1 + \beta_p(t)\right) - 1\right| \quad (n < m).$$

For $|x| < 1/2$, we have $\mathrm{e}^{x-x^2} \leqslant 1 + x \leqslant \mathrm{e}^x$. Thus we obtain $q_A^n(t) \leqslant \exp\{\sum_{p=1}^{n}\beta_p(t)\}$ and

$$\exp\left\{\sum_{p=n+1}^{m}\beta_p(t) - \sum_{p=n+1}^{m}\beta_p^2(t)\right\} \leqslant \prod_{p=n+1}^{m}\left(1 + \beta_p(t)\right) \leqslant \exp\left\{\sum_{p=n+1}^{m}\beta_p(t)\right\}.$$

Here, $\sum_{p=n+1}^{m}\beta_p^2(t) \leqslant Cn^{-1}$, due to $\beta_p^2(t) \leqslant Cp^{-2}$. Moreover, using a summation by parts, we get

$$\sum_{p=n+1}^{m}\beta_p(t) = \frac{D_{n,m}(t)}{\bar{d} + m - 1} + \sum_{p=n+1}^{m-1}\frac{D_{n,p}(t)}{(\bar{d} + p - 1)(\bar{d} + p)} \quad \text{with } D_{n,m}(t) = \sum_{p=n+1}^{m}(d_{t-p} - \bar{d}). \tag{9}$$

The hypothesis (6) implies that $|D_{n,m}(t)| \leqslant C|n-m|^{1-\delta}$ where $C$ does not depend on $t$. Thus, it is easy to obtain that the r.h.s. of (9) does not exceed $Cn^{-\delta}$. Therefore, for all $n$ sufficiently large,

$$\mathrm{e}^{-Cn^{-\delta}} \leqslant \prod_{p=n+1}^{m} \big(1+\beta_p(t)\big) \leqslant \mathrm{e}^{Cn^{-\delta}}, \tag{10}$$

for some constant $C>0$ independent of $n$ and $m>n$. It also follows that $\sup_{n\geqslant 1,\, t\in\mathbb{Z}} q_A^n(t) < \infty$. Finally, the class $\mathcal{M}$ is closed under algebraic operations, shifts and uniform limits, therefore $q_A \in \mathcal{M}$.

The proof for $q_B(t)$ follows similarly. □

In this lemma, we establish some relations between the time varying coefficients and the fractional coefficients defined in (2).

**Lemma 2.3.** *Under the hypotheses of Lemma* 2.2*,*

$$a^-(s,t) = q_A(t)\psi_{t-s}(-\bar{d})\theta_{s,t}, \qquad b^-(s,t) = q_B(s)\frac{d_{t-1}}{\bar{d}}\psi_{t-s}(-\bar{d})\vartheta_{s,t},$$

*where* $|\theta_{s,t}-1| + |\vartheta_{s,t}-1| \leqslant C|s-t|^{-\delta}$ $(s<t)$*, with some constant* $C<\infty$ *independent of* $s,t$*.*

**Proof.** By definition, we have

$$\theta_{s,t} = \frac{a^-(s,t)}{q_A(t)\psi_{t-s}(-\bar{d})} = \prod_{p=1}^{\infty}\big(1+\beta_p(t)\big)^{-1},$$

where the $\beta_p(t)$'s are defined in (8). From inequality (10), one has $\mathrm{e}^{-C(t-s)^{-\delta}} \leqslant \theta_{s,t} \leqslant \mathrm{e}^{C(t-s)^{-\delta}}$ for all $s<t$ with $C>0$ independent of $(s,t)$. This yields $|\theta_{s,t}-1| \leqslant C|s-t|^{-\delta}$ for all $s<t$.

The proof for $b^-(s,t)$ and $\vartheta_{s,t}$ follows similarly. □

## 3. Main result

In this section, we prove that the time varying processes defined in (4) and (5) satisfy an invariance principle. It is important to notice that we obtain the same limit as for the partial sums of a fractional process $Z_t = (I-B)^{-d}\varepsilon_t$ (see [5]).

**Theorem 3.1.** *If we assume that the sequence* $\mathbf{d} \in \mathcal{M}$ *satisfies the condition* (6) *and its mean value* $\bar{d} \in (0,1/2)$*. Then the partials sums of the processes* $(X_t)$ *and* $(Y_t)$ *defined in* (4) *and* (5) *weakly converge in the Skorohod space* $D[0,1]$*:*

$$N^{-\bar{d}-(1/2)}\sum_{t=1}^{[N\tau]} Y_t \xrightarrow{D[0,1]} c_A B_H(\tau) \quad\text{and}\quad N^{-\bar{d}-(1/2)}\sum_{t=1}^{[N\tau]} X_t \xrightarrow{D[0,1]} c_B B_H(\tau), \tag{11}$$

*where* $B_H$ *is the standard fractional Brownian motion with Hurst parameter* $H = 1/2+\bar{d}$ *and variance* $EB_H^2(1)=1$*. The positive constants* $c_A$ *and* $c_B$ *are given by*

$$c_A^2 := \frac{\overline{q_A}^2}{\bar{d}(1+2\bar{d})}\frac{\Gamma(1-2\bar{d})}{\Gamma(1-\bar{d})\Gamma(\bar{d})} \quad \text{and} \quad c_B^2 := \frac{\overline{q_B^2}}{\bar{d}(1+2\bar{d})}\frac{\Gamma(1-2\bar{d})}{\Gamma(1-\bar{d})\Gamma(\bar{d})}, \tag{12}$$

*where* $\overline{q_A}$ *and* $\overline{q_B^2}$ *are the mean values of the averageable sequences* $q_A$ *and* $q_B^2$ *defined in* (7)*.*

**Proof.** We prove the convergence of finite dimensional distributions from the so-called scheme of discrete stochastic integrals [4]. We restrict the proof to the case $\tau=1$. For $s>t$, we put $a^-(s,t) = b^-(s,t) := 0$.

Firstly, the limit $c_A B_H(1)$ can be represented as a stochastic integral $J := \int_{\mathbb{R}} f(x)\,\mathrm{d}Z(x)$ with respect to a standard Gaussian white noise $Z$ and $f(x) := \frac{c_A}{\nu(\bar{d})}\int_0^1 (t-x)_+^{\bar{d}-1}\,\mathrm{d}t\,\mathbb{I}_{]-\infty,1]}(x)$. See [5] for detail on the representation of $B_H$ as a stochastic integral and an explicit form of the positive constant $\nu(d)$, $\nu(\bar{d})^2 = \int_{\infty}^{1}(\int_0^1 (t-x)_+^{\bar{d}-1}\,\mathrm{d}t)^2\,\mathrm{d}x$.

Secondly, the normalized partial sums of $(Y_t)$ can be written as discrete stochastic integrals. Taking $\tau = 1$ this integral is $J_N := \int f_N(x)\,\mathrm{d}Z_N(x)$, where $Z_N$ is a 'discrete stochastic measure' defined by

$$Z_N(x', x''] := N^{-1/2} \sum_{x'N < s \leqslant x''N} \varepsilon_s \quad \text{and} \quad f_N(x) := N^{-\bar{d}} \sum_{s=-\infty}^{N} \sum_{t=1}^{N} a^-(s,t) \mathbb{I}_{]\frac{s-1}{N}, \frac{s}{N}]}(x).$$

The convergence in distribution of $J_N$ to $J$ follows from (see [4] for details)

(i) $(Z_N(x_1', x_1''], \ldots, Z_N(x_m', x_m'']) \overset{\text{law}}{\to} (Z(x_1', x''], \ldots, Z(x_m', x_m''])$ for any $m < \infty$ and any disjoint intervals $(x_i', x_i''], i = 1, \ldots, m$;
(ii) $\|f_N - f\|_2 \to 0$, in the Hilbert space $L^2(\mathbb{R})$.

Step (i) is immediate by the central limit theorem. To prove the step (ii), using Lemma 2.3, it suffices to show the convergence $\|\tilde{f}_N - f\|_2 \to 0$, where

$$\tilde{f}_N(x) := N^{-\bar{d}} \sum_{s=-\infty}^{N} \sum_{t=1}^{N} q_A(t) \psi_{t-s} \mathbb{I}_{]\frac{s-1}{N}, \frac{s}{N}]}(x),$$

where $\psi_{t-s} = \psi_{t-s}(-\bar{d})$ is the fractional coefficient defined in (2). Next, we write $\tilde{f}_N = f_N' + f_N''$, where

$$f_N'(x) := N^{-\bar{d}} \bar{q}_A \sum_{s=-\infty}^{N} \sum_{t=1}^{N} \psi_{t-s} \mathbb{I}_{]\frac{s-1}{N}, \frac{s}{N}]}(x), \qquad f_N''(x) := N^{-\bar{d}} \sum_{s=-\infty}^{N} \sum_{t=1}^{N} \big(q_A(t) - \bar{q}_A\big) \psi_{t-s} \mathbb{I}_{]\frac{s-1}{N}, \frac{s}{N}]}(x).$$

The convergence $\|f_N' - f\|_2 \to 0$ is well-known (see [4]). Relation $\|f_N''\|_2 \to 0$ can be shown using an integration by parts and the fact $q_A(t) - \overline{q_A}$ is averageable with zero mean. This proves the convergence of finite dimensional distribution of partial sums of $(Y_t)$.

The tightness in $D[0,1]$ is obtained by the criterion given in [1] (Theorem 15.6). Using Lemma 2.3 and the boundedness of $q_A(t)$, the proof is standard. This proves the convergence in (11) for $(Y_t)$.

The proof for $(X_t)$ is similar. The major modification is to replace $Z_N$ by

$$\widetilde{Z}_N(x', x''] := \big(\overline{q_B^2}\big)^{-1/2} N^{-1/2} \sum_{x'N < s \leqslant x''N} q_B(s) \varepsilon_s.$$

In this case, the central limit theorem (step (i)) follows by independence of $(\varepsilon_t,\ t \in \mathbb{Z})$ and the fact that

$$E\big(\widetilde{Z}_N(x', x'']\big)^2 = \big(\overline{q_B^2}\big)^{-1} N^{-1} \sum_{x'N < s \leqslant x''N} q_B^2(s) \to x'' - x',$$

because $q_B \in \mathcal{M}$, and so $(q_B^2(s),\ s \in \mathbb{Z})$ is averageable. The remaining details are the same as in the case of $(Y_t)$. □

## References

[1] P. Billingsley, Convergence of Probability Measures, John Wiley & Sons Inc., New York, 1968.
[2] P.J. Brockwell, R.A. Davis, Time Series: Theory and Methods, second ed., Springer-Verlag, New York, 1991.
[3] A. Philippe, D. Surgailis, M.-C. Viano, Time-varying fractionally integrated processes with nonstationary long memory, Technical report, Pub. IRMA Lille, 61(9), 2004.
[4] D. Surgailis, Non-CLTs: $U$-statistics, multinomial formula and approximations of multiple Itô-Wiener integrals, in: P. Doukhan, et al. (Eds.), Theory and Applications of Long-Range Dependence, Birkhäuser, Boston, MA, 2003, pp. 129–142.
[5] M.S. Taqqu, Fractional Brownian motion and long-range dependence, in: P. Doukhan, et al. (Eds.), Theory and Applications of Long-Range Dependence, Birkhäuser, Boston, MA, 2003, pp. 5–38.