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Relative hyperbolization and Pontrjagin classes

B.Z. Hu

Department of Mathematics, State University of New York at Binghamton, Binghamton, NY 13902, USA

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Presented by Miklaël Gromov

Abstract

The 'nonpositive curvature retraction theorem' has a refined version, a version that respects characteristic classes. ***To cite this article: B.Z. Hu, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Hyperbolization relatif et classes de Pontrjagin. Nous précisons que le « théorème de rétraction en courbure non-positif » à une version plus raffiné, une version qui respecte les classes caractéristiques. ***Pour citer cet article : B.Z. Hu, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Un sous-espace $Y \subset X$ est une rétraction s'il existe une application $R : X \to Y$ tels que $R|_Y = \mathrm{Id}_Y$. Le « théorème de la rétraction géométrique » ([5] 2.4) indique que n'importe quel polyèdre compact K de courbure $\leqslant 0$ peut être exprimé comme une rétraction d'une variété fermée semi-linéaire M de courbure $\leqslant 0$. Dans cette Note nous précisons une version plus raffiné de ce résultat :

Théorème 0.1. *Supposons que K est un polyèdre compact de courbure $\leqslant 0$, E un fibré vectorielle sur K. Il éxiste alors une variété fermé semi-linéaire M de courbure $\leqslant 0$, tels que $K \subset M$ est une rétraction, et ayant la propriété que $T(M)|_K \overset{stablement}{=} E$, où $T(M)$ est le fibré vectorielle tangent de M.*

Esquisse de la démonstration du Théorème 0.1. Comme pour la version originale, nous démontrons le Théorème 0.1 en employant le procédeé d'hyperbolization relatif, une notion définis pour la première fois dans [3]. Nous emploierons le procédé d'hyperbolization relatif « h » décrit dans [5] §§ 1–2, et la construction « p » décrit dans [5] § 3. Supposons que K est un polyèdre compact équipé d'une metrique Euclidienne par morceaux, de courbure $\leqslant 0$, et que E est un fibré vectorielle sur K. Exprimons K comme plein sous-complex d'une variété triangulée fermée P. Par le théorème de la rétraction géométrique, on obtient une variété triangulée fermée $h(P, K)$ de courbure $\leqslant 0$, et une

E-mail address: zgfjqz@hotmail.com (B.Z. Hu).

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copie homéomorphe $K \subset h(P,K)$ qui est une rétraction de $h(P,X)$. Prolongons le fibré vectorielle E définis sur K à la variété $h(P,K)$. Choisissons maintenant un fibré vectorielle F sur $h(P,K)$ tels que $T(h(P,K)) \oplus F \overset{\text{stablement}}{=} E$, où $T(h(P,K))$ est le fibré vectorielle tangent de $h(P,K)$. Laissons Q être un voisinage régulier de $h(P,K) \subset F$, et $Q \cup Q$ le double de Q le long de sa frontière. Prenons $M = h(Q \cup Q, h(P,K))$; comme $h(P,K)$ est une variété de courbure $\leqslant 0$, M est également une variété fermée semi-linéaire de courbure $\leqslant 0$, et K est une rétraction de M. Laissons $N = p(Q \cup Q, h(P,K))$; la réstriction de $T(M)$ sur N est $T(N)$, et $T(N)$ est équivalent à $T(Q)$ sur la sous-variété $h(P,K)$. De plus, sur la sous-variété $h(P,K)$, $T(Q)$ est la somme du fibré vectorielle F avec le fibré vectorielle tangent de $h(P,K)$. Par conséquent $T(M) \overset{\text{stablement}}{=} E$ sur la sous-variété $h(P,K)$, ce qui permet de conclure que $T(M)|_K \overset{\text{stablement}}{=} E$. □

Théorème 0.2. *Supposons que K est un polyèdre asphérique compact, E un fibré vectorielle sur K. Il éxiste alors une variété asphérique fermée M, tels que K est une rétraction de M, et sur laquelle $T(M)_K \overset{\text{stablement}}{=} E$, où $T(M)$ est le fibré vectorielle tangent de M.*

Le « théorème de la rétraction asphérique » de Davis déclare que n'importe quel polyèdre asphérique compact peut être éxprimée comme une rétraction d'une variété asphérique fermée (voir, par exemple, [1]). Ainsi, le Théorème 0.2 est une version plus raffiné de ce resultat. La preuve du Théorème 0.2 emploie également le procedé d'hyperbolization relatif. L'hyperbolization relatif peut être appliqué dans le contexte des polyèdres asphériques, car il existe une version asphérique du lemme de collage totalement géodésique de Gromov.

Corollaire 0.3. *Pour tout polyèdre compact K de courbure $\leqslant 0$, et tout $\mathcal{C} \in \bigoplus_{i \geqslant 1} H^{4i}(K,\mathbb{Q})$, il existe une variété fermée semi-linéaire M de courbure $\leqslant 0$, tels que $K \subset M$ est une rétraction, et que*

$$\left(p_1(M) + \frac{p_1^2(M) - 2p_2(M)}{12} + \cdots\right)\bigg|_K = \mathcal{C} \times n$$

où $ph = \dim + p_1 + (p_1^2 - 2p_2)/12 + \cdots$ est le caractère de Pontrjagin, et n est un certain entier positif qui dépend de C.

1. Introduction

A subspace $Y \subset X$ is a retraction if there exists a map $R : X \to Y$ such that $R|_Y = \mathrm{Id}_Y$. The existence of a retraction is a very strong topological condition in that it tells that no topological property of Y will ever be lost in X. It does not matter whether this is a soft topological property involving, e.g., homology/homotopy groups, or a hard topological property involving, e.g., Whitehead/Surgery groups. Recall that the 'geometric retraction theorem' ([5] 2.4) says that any compact polyhedron K with curvature $\leqslant 0$ is the retraction of a closed PL manifold M with curvature $\leqslant 0$. In this Note we point out that this result has a refined version as follows:

Theorem 1.1. *Suppose K is a compact polyhedron with curvature $\leqslant 0$, E is a vector bundle over K. Then there exists a closed PL manifold M with curvature $\leqslant 0$, such that $K \subset M$ is a retraction, and that $T(M)|_K \overset{\text{stably}}{=} E$, where $T(M)$ is the tangent bundle of M.*

Like the original version, Theorem 1.1 will also be proven via relative hyperbolization, a concept first appeared in [3]. The relative hyperbolization 'h' as described in [5] §§1–2 (see also [4] 2.2 and [2] 4a) is one of the rare relative hyperbolization that is *universally good*, and is the one that will be used here. There is another construction 'p' as described in [5] §3, which will also be used here. Suppose X is a cell complex, $K \subset X$ is subcomplex. To construct $h(X,K)$ we start with vertices, select only those that are in K and discard the others, so we get K^0. Next, start with two copies of K^0, $K^0 \times (\pm 1)$, and consider 1-cells; if a 1-cell Δ^1 does not intersect K, it is discarded, if $\Delta^1 \subset K$, simply add $\Delta^1 \times (\pm 1)$ to $K^0 \times (\pm 1)$, if $\Delta^1 \cap K = V$ is 1 or 2 vertices, add $V \times [-1,1]$ to $K^0 \times (\pm 1)$, call the result $h(X^1, X^1 \cap K)$. Then, start with two copies of $h(X^1, X^1 \cap K)$, $h(X^1, X^1 \cap K) \times (\pm 1)$, etc. The final result is our $h(X,K)$, the strict definition of which has to involve axioms. The construction of $p(X,K)$ is similar, except that we do not double, and we glue only along one side.

Two bundles are said to be stably isomorphic if they are isomorphic after adding trivial bundles. Note that we may state two versions of Theorem 1.1, one in terms of actual vector bundles, another in terms of block bundles, both versions are valid, their proofs completely parallel. Suppose X is a combinatorial manifold, $K \subset X$ is subcomplex, then $h(X,K)$ is a combinatorial manifold. To work in the category of vector bundles, we need $h(X,K)$ to have smooth structure. For this, embed K in an X whose PL structure can be smoothed, e.g., a sphere or Euclidean space. If the PL structure of X can be smoothed, then so can that of $h(X,K)$, which is one of the consequences of the technique called 'bending'. To work in the category of block bundles, any X will do. For block bundles, their operations, tangent/normal block bundles, etc. consult [7], and [8] §5.

Suppose X is a finite simplicial complex, K is full subcomplex, i.e., for any simplex $\Delta \subset X$, $\Delta \cap K$ is empty or simplex. Note that if a subcomplex of a cell complex is not already full, you can carry out a barycentric subdivision to make it full. Recall that the regular neighborhood of $K \subset X$, here denoted as $r(X,K)$, is defined by carrying out barycentric subdivision of X and letting $r(X,K)$ be the union of the barycentric simplices intersecting K. One may want to call $r(X,K)$ the cone-like regular neighborhood and $p(X,K)$ the cube-like regular neighborhood. It turns out that, although $p(X,K)$ is not naturally situated in X, it is more useful than the standard $r(X,K)$ when geometry is involved. $p(X,K)$ is the one to use in Euclidean geometry while $r(X,K)$ is the one to use in spherical geometry.

Lemma 1.2. *Suppose X is a finite simplicial complex, K is full subcomplex. Then there is a PL isomorphism between $p(X,K)$ and $r(X,K)$, which is identity on K, such that for any subcomplex $Y \subset X$, the isomorphism between $p(X,K)$ and $r(X,K)$ restricts to the isomorphism between $p(Y, Y\cap K)$ and $r(Y, Y\cap K)$.*

Proof of Lemma 1.2. Note that this lemma does not involve geometry, also, one is forced to proving a statement more general than needed due to the way 'p' is defined. We prove via induction on the relative dimension $\dim(X,K)$. If $\dim(X,K)=0$, i.e., X is K plus vertices, then both $p(X,K)$ and $r(X,K)$ are K, so the combinatorial equivalence is simply the identity of K. In case $\dim(X,K)=1$, X is the union of K with line segments a side of which are in K, so $p(X,K)$ is the union of K and those vertices crossed with $[0,1]$, while $r(X,K)$ is the union of K and the halves of those line segments, between the two of which obviously exists a PL isomorphism. (The reader may want to have a look at the relative dimension 2 case.) In general, suppose $\dim(X,K)=n$, and the statement on 'p' and 'r' has been verified for $\leqslant n-1$ cases. Consider any simplex Δ^n of X. $\Delta^n \cap K$ is ϕ or simplex. If $\Delta^n \cap K = \phi$, both $p(\Delta^n,\phi)$ and $r(\Delta^n,\phi)$ are ϕ. Suppose $\Delta^n \cap K = \Delta^i$, $i \leqslant n$. If $i=n$, both $p(\Delta^n,\Delta^n)$ and $r(\Delta^n,\Delta^n)$ are Δ^n. Suppose $i<n$. Now, $p(\Delta^n,\Delta^i) = p(\partial\Delta^n,\Delta^i)\times[0,1]$, while $r(\Delta^n,\Delta^i) = \mathrm{Cone}(r(\partial\Delta^n,\Delta^i))$, where $r(\partial\Delta^n,\Delta^i)$ is the regular neighborhood of $\Delta^i \subset \partial\Delta^n$, i.e., the union of the barycentric simplices in $\partial\Delta^n$ that intersect Δ^i, and where the cone vertex is the barycenter of Δ^n. By induction hypothesis, there is a natural PL isomorphism between $p(\partial\Delta^n,\Delta^i)$ and $r(\partial\Delta^n,\Delta^i)$ which is identity on Δ^i. Note that $r(\partial\Delta^n,\Delta^i)$ is a combinatorial $(n-1)$-disk, so the PL isomorphism between $p(\partial\Delta^n,\Delta^i)$ and $r(\partial\Delta^n,\Delta^i)$ can be extended to one between $p(\partial\Delta^n,\Delta^i)\times[0,1]$ and $\mathrm{Cone}(r(\partial\Delta^n,\Delta^i))$, i.e., between $p(\Delta^n,\Delta^i)$ and $r(\Delta^n,\Delta^i)$, by appealing to Alexander's trick twice. (It should also be possible to actually construct an extension). Now, $p(X,K)=\bigcup_{\Delta^i\subset X} p(\Delta^i,\Delta^i\cap K)\times 0^{n-i}$, $r(X,K)=\bigcup_{\Delta^i\subset X} r(\Delta^i,\Delta^i\cap K)$, the natural isomorphism between $p(\Delta^i,\Delta^i\cap K)$ and $r(\Delta^i,\Delta^i\cap K)$ together form a natural isomorphism between $p(X,K)$ and $r(X,K)$. End of proof of Lemma 1.2. □

2. Proof of Theorem 1.1

Suppose K is a compact polyhedron with piecewise Euclidean structure with curvature $\leqslant 0$, E is a vector bundle over K. As usual, we may assume that P is a closed triangulated manifold, $K \subset P$ is full subcomplex. Clarifying a point here: when applying relative hyperbolization to (P,K), you do not really need PL geometry on P if you are using the 'h' of [5], although you do if you use the 'h' of [4]. By the geometric retraction theorem ([5] 2.4), $h(P,K)$ is a closed triangulated manifold with curvature $\leqslant 0$ and $K \subset h(P,K)$ is a retraction. Since $K \subset h(P,K)$ is a retraction, the vector bundle E over K can be extended to one over $h(P,K)$, still denoted as E. Now choose a vector bundle F over $h(P,K)$ such that $T(h(P,K)) \oplus F \overset{\text{stably}}{=} E$, where $T(h(P,K))$ is tangent bundle of $h(P,K)$. Consider $h(P,K)\subset F$, let Q be a regular neighborhood for this, let $Q\cup Q$ be the double along boundary. Apply relative hyperbolization to $h(P,K)\subset Q\cup Q$, let $M=h(Q\cup Q, h(P,K))$, M is closed PL manifold with curvature $\leqslant 0$ because $h(P,K)$ satisfies curvature $\leqslant 0$. Since K is retraction of $h(P,K)$, $h(P,K)$ is retraction of M, so K is retraction of M. Now consider $T(M)$ over K. Let $N=p(Q\cup Q, h(P,K))$, $N\subset M$ is a codimension 0 submanifold, because M and N are of the same dimension as cell complexes, $N \subset M$ is subcomplex, M is combinatorial manifold, while N is combinatorial manifold with boundary according to [5] 3.6 & 3.4(3). So the restriction of $T(M)$ over N is $T(N)$. By Lemma 1.2, N is equivalent to the regular neighborhood of $h(P, K) \subset Q$, so $T(N)$ is equivalent to $T(Q)$ over $h(P, K)$. Consider $T(Q)$ over $h(P, K)$, it is simply the tangent bundle of $h(P, K)$ plus the normal bundle of $h(P, K) \subset F$. The normal bundle of $h(P, K) \subset F$ is F itself. $T(h(P, K)) \oplus F \overset{\text{stably}}{=} E$. Therefore, $T(M) \overset{\text{stably}}{=} E$ over $h(P, K)$. Therefore $T(M)|_K \overset{\text{stably}}{=} E$. This proves Theorem 1.1.

Note that my original purpose was to obtain Theorem 1.1 in case E is a line bundle, which is a non-Riemannian version of [6] p. 259, be needed.

3. Further remarks, and consequences of Theorem 1.1

Remark 1. The 'fundamental group surrounding relation' states that, the fundamental group of a compact polyhedron with curvature $\leqslant 0$ is 'surrounded' by a finite number of fundamental groups of closed PL manifolds with curvature $\leqslant 0$, see [4] 2.6 and [5] 2.1. We point out that this relation can also be refined.

Theorem 3.1. *Suppose K is a compact aspherical polyhedron, E is a vector bundle over K. Then there exists a closed aspherical manifold M, such that K is a retraction of M, and such that $T(M)|_K \overset{\text{stably}}{=} E$, where $T(M)$ is the tangent bundle of M.*

Sketch of proof of Theorem 3.1. The 'aspherical retraction theorem' of Davis says that any compact aspherical polyhedron is the retraction of a closed aspherical manifold, which was first proven using the reflection group trick, see, e.g., [1]. As discussed in [5], the construction 'h' applies equally well to aspherical polyhedra without geometry, and can yield a different proof of Davis's theorem. The reason is simply that, when it comes to the construction 'h', in dealing with piecewise Euclidean geometry, we only use Gromov's 'totally geodesic gluing lemma' without ever gotten involved in geometric links, and the gluing lemma has an easy aspherical counterpart, which I forgot to clarify in [5]. (Aspherical counterpart of Gromov's gluing lemma: Assume that P and Q are aspherical polyhedra, that their intersection is an aspherical subpolyhedron A, and that $A \subset P$, and $A \subset Q$ are π_1-injective, then $P \cup Q$ is aspherical and $P \subset P \cup Q$ and $Q \subset P \cup Q$ are π_1-injective. Compare [4] 2.3.1.) Now if one goes through the proof of Theorem 1.1, one sees that it can be obviously adjusted to proving Theorem 3.1 (details are left to the reader). □

Remark 2. If you go through Davis's 'reflection group trick' instead, you will see that you can also prove Theorem 3.1 by doing the same as in proof of Theorem 1.1 but simply using that trick to replace our 'h' in the process. On the other hand, there is the issue of whether you can use the reflection group trick to prove the geometric retraction theorem ([5] 2.4) and its refined version (Theorem 1.1 above). As suggested by [1] §17, one would like to glue together copies of 'p' using simple reflection groups. This requires understanding 'p' and 'δ'([5] 3.5) and their geometric links. My proposed approach to investigating the geometric links of 'p' (and also of 'h') is as follows: You define a process which I called 'spherical relative hyperbolization'. Then you prove that the geometric links of the 'Euclidean relative hyperbolization', i.e., 'h', are mostly just the 'spherical relative hyperbolization'. Finally you establish a result on 'spherical relative hyperbolization' which is kind of a partial relative version of the so-called 'no-Δ lemma' of Gromov.

Recall that an absolute hyperbolization is a process that changes a compact polyhedron X to $\widetilde{X}$ to gain curvature $\leqslant 0$. In contrast to relative hyperbolization, during the absolute hyperbolization process, most topological properties of X, except homological ones, are lost in $\widetilde{X}$. But when X is a closed PL manifold, there is usually a map $f : \widetilde{X} \to X$ that pulls back the Pontrjagin classes of X to those of $\widetilde{X}$, because it pulls back the stable tangent bundle of X to that of $\widetilde{X}$, as originally pointed out in [3] and [2]. Theorem 1.1 can be regarded as the relative version of this feature:

Corollary 3.2. *For any compact polyhedron K with curvature $\leqslant 0$, any $\mathcal{C} \in \bigoplus_{i \geqslant 1} H^{4i}(K, \mathbb{Q})$, there exists a closed PL manifold M with curvature $\leqslant 0$, such that $K \subset M$ is a retraction, and that*

$$\left(p_1(M) + \frac{p_1^2(M) - 2p_2(M)}{12} + \cdots \right)\bigg|_K = \mathcal{C} \times n$$

where $ph = \dim + p_1 + (p_1^2 - 2p_2)/12 + \cdots$ is the Pontrjagin character, n is a certain positive integer depending on C.

Reason. Let $KO()$ be the real K-theory, the Pontrjagin character $ph : KO(K) \to \bigoplus_{i \geqslant 0} H^{4i}(K, \mathbb{Q})$ is an isomorphism, which follows quite trivially from that the Chern character is isomorphism. So $\mathcal{C}$ can be expressed as $(ph(E) - \dim(E))/n$, where E is a vector bundle over K while n is a positive integer. □

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