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Topology

Steenrod squares on conjugation spaces

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Abstract

We prove that the coefficients of the so-called conjugation equation for conjugation spaces in the sense of Hausmann–Holm–Puppe are completely determined by Steenrod squares. This generalises a result of V.A. Krasnov for certain complex algebraic varieties. It also leads to a generalisation of a formula given by Borel and Haefliger, thereby largely answering an old question of theirs in the affirmative. ***To cite this article: M. Franz, V. Puppe, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Carrés de Steenrod dans les espaces avec conjugaison. On démontre que les coefficients de l'équation dite « de conjugaison », pour les espaces avec conjugaison au sens de Hausmann–Holm–Puppe, sont complètement déterminés par les carrés de Steenrod. Ceci généralise un résultat de V. A. Krasnov sur certaines variétés algébriques complexes, ainsi qu'une formule de Borel–Haefliger, donnant ainsi une réponse positive à une question de ces deux derniers auteurs. ***Pour citer cet article : M. Franz, V. Puppe, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Statement of the results

Let X be a topological space with an involution τ. We look at X as a space with an action of the group $G=\{1,\tau\}$. We take cohomology with coefficients in $\mathbf{F}_2$ and consider the restriction map $r: H^*(X)\to H^*(X^\tau)$, its equivariant counterpart $r_G: H^*_G(X)\to H^*_G(X^\tau)=H^*(X^\tau)\otimes H^*(BG)$ and the canonical projection $p: H^*_G(X)\to H^*(X)$. Recall that $H^*(BG)=H^*(\mathbf{RP}^\infty)=\mathbf{F}_2[u]$ with $\deg(u)=1$.

According to Hausmann–Holm–Puppe [2], (X,τ) is called a *conjugation space* if $H^{\mathrm{odd}}(X)=0$ and if there exists a section $\sigma: H^*(X)\to H^*_G(X)$ of p and a degree-halving isomorphism $\kappa: H^{2*}(X)\to H^*(X^\tau)$ with the following property: for every $x\in H^{2n}(X)$, $n\in\mathbf{N}$, there exists elements $y_1,\dots,y_n\in H^*(X^\tau)$ such that the so-called *conjugation equation* holds:

$$r_G\bigl(\sigma(x)\bigr)=\kappa(x)u^n+y_1u^{n-1}+\cdots+y_{n-1}u+y_n. \tag{1}$$

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A priori, σ and κ are only assumed to be additive, but the conjugation equation implies that they are in fact multiplicative and unique. There are many examples of conjugation spaces, including flag manifolds, co-adjoint orbits of compact Lie groups and (compact) toric manifolds, see [2].

For the conjugation space $\mathbf{CP}^k$, $k \leq \infty$, Hausmann–Holm–Puppe prove the formula $r_G(\sigma(v^n)) = (wu + w^2)^n$, where $v \in H^2(\mathbf{CP}^k)$ and $w \in H^1(\mathbf{RP}^k)$ denote the generators [2, Example 3.7]. In other words, $r_G(\sigma(v^n)) = (wu + \mathrm{Sq}^1(w))^n$. The following result generalises this to an arbitrary conjugation space X.

Theorem 1.1. *For every $x \in H^{2n}(X)$, $n \in \mathbf{N}$, one has*

$$r_G\bigl(\sigma(x)\bigr) = \sum_{i=0}^{n} \mathrm{Sq}^i\bigl(\kappa(x)\bigr)u^{n-i} =: \mathrm{SQ}\bigl(\kappa(x)\bigr).$$

Corollary 1.2. *For every $x \in H^*(X)$, one has*

$$r(x) = \kappa(x)^2.$$

We also show that the isomorphism κ commutes with total Steenrod squares.

Theorem 1.3. *For every $x \in H^*(X)$ one has*

$$\kappa\bigl(\mathrm{Sq}(x)\bigr) = \mathrm{Sq}\bigl(\kappa(x)\bigr).$$

Note that the odd Steenrod squares of x vanish since $H^*(X)$ is concentrated in even degrees. Hence, the above identity is equivalent to

$$\kappa\bigl(\mathrm{Sq}^{2k}(x)\bigr) = \mathrm{Sq}^k\bigl(\kappa(x)\bigr) \quad \text{for all } k \in \mathbf{N}. \tag{2}$$

2. Proofs

We denote the Steenrod algebra for the prime 2 by $\mathcal{A}$.

Lemma 2.1. *For every n there exist universal elements $a_0, \ldots, a_n, b \in \mathcal{A}$ such that for every conjugation space X and every $x \in H^{2n}(X)$ one has*

$$r_G\bigl(\sigma(x)\bigr) = \sum_{i=0}^{n} a_i\bigl(\kappa(x)\bigr)\, u^{n-i} \quad \textit{and} \quad \kappa\bigl(\mathrm{Sq}(x)\bigr) = b\bigl(\kappa(x)\bigr).$$

Moreover, $a_0 = 1$ and $a_1 = \mathrm{Sq}^1$.

Proof. Since κ is bijective, one can define, for every X, functions $a_i, b : H^*(X) \to H^*(X^\tau)$ such that the above identities hold. We show that they are (or, more precisely, come from) Steenrod squares, using that the restriction map r_G commutes with all Steenrod squares. We write $\kappa(x) = z$.

We start by proving the claim about the a_i by induction on i, beginning at $a_0(z) = z$. If $i > 0$ is even, we apply Sq^{2k}, where $k \leqslant i/2$ will be chosen later. By the Leray–Hirsch theorem, we can write

$$\mathrm{Sq}^{2k}\bigl(\sigma(x)\bigr) = \sum_{l=-n}^{k} \sigma(x_l)\, u^{2(k-l)} \tag{3}$$

for some $x_l \in H^{2(n+l)}(X)$. Write $z_l = \kappa(x_l)$. The restriction $r_G(\sigma(x_l)u^{2(k-l)})$ has leading term $z_l u^{n+2k-l}$, while, by (1) and the Cartan formula, the leading power of u in $\mathrm{Sq}^{2k}(r_G(\sigma(x)))$ is at most u^{n+2k}. Hence, the summation in (3) is in fact only over $0 \leqslant l \leqslant k$.

We first compare coefficients of u^{n+2k-l} in $r_G(\mathrm{Sq}^{2k}(\sigma(x))) = \mathrm{Sq}^{2k}(r_G(\sigma(x)))$. Using Eq. (3) and the formula [5, Lemma 2.4]

$$\mathrm{Sq}^j\bigl(u^i\bigr) = \binom{i}{j} u^{i+j},$$

we get for $0 \leqslant l \leqslant k$

$$z_l = \sum_{j=0}^{l} \binom{n-l+j}{2k-j} \mathrm{Sq}^j\big(a_{l-j}(z)\big) + \sum_{j=1}^{l} a_j(z_{l-j}), \tag{4}$$

in particular

$$z_0 = \binom{n}{2k} z.$$

Since $l < i$, this inductively shows $z_l = b_l(z)$ for some $b_l \in \mathcal{A}$. Comparing coefficients of u^{n+2k-i} then gives

$$\sum_{l=0}^{k} a_{i-l}(z_l) = \binom{n}{2k} a_i(z) + \sum_{l=1}^{k} a_{i-l}\big(b_l(z)\big) = \binom{n-i}{2k} a_i(z) + \sum_{j=1}^{2k} \binom{n-i+j}{2k-j} \mathrm{Sq}^j\big(a_{i-j}(z)\big). \tag{5}$$

Now suppose that $k \leqslant i/2$ is such that

$$\binom{n}{2k} \neq \binom{n-i}{2k}.$$

For instance, this is true if $2k$ is the largest power of 2 dividing i. (Recall that a binomial coefficient mod 2 is the product of the binomial coefficients taken for each pair of binary digits, cf. [5, Lemma I.2.6].) Then Eq. (5) can be solved for $a_i(z)$ and shows that $a_i(z)$ can be expressed in terms of repeated Steenrod squares of z.

For odd i, a similar (but simpler) reasoning based on commutativity with respect to Sq^1 gives $a_i(z) = \mathrm{Sq}^1(a_{i-1}(z))$, in particular $a_1(z) = \mathrm{Sq}^1(z)$.

Now that all $a_i(z)$ are known, we apply Sq^{2k} for any k. Using the same notation as above, we have $\mathrm{Sq}^{2k}(x) = p(\mathrm{Sq}^{2k}(\sigma(x))) = x_k$. Comparing coefficients as before gives a formula for $b_l(z)$ similar to Eq. (4), but where the summation index j starts at $l-n$ if $l > n$. Still, the equations can be recursively solved for z_l. Hence,

$$\kappa\big(\mathrm{Sq}(x)\big) = \kappa(x_0) + \cdots + \kappa(x_n) = b_0(z) + \cdots + b_n(z) = b(z). \qquad \square$$

In principle, the preceding proof could be used to determine the coefficients of the conjugation equation completely (as well as those of $\mathrm{Sq}^k(\sigma(x))$ for any k). We will take a less tedious approach which relies on the fact that suitable products of infinite-dimensional real projective space can "detect" Steenrod squares, cf. [5, Corollary I.3.3].

Fact 2.2. *The restricted evaluation map $\mathcal{A}_{\leqslant n} \to H^*((\mathbf{RP}^\infty)^n)$, $a \mapsto a(w \times \cdots \times w)$ is injective for any $n \in \mathbf{N}$.*

Proof of Theorem 1.1. We want to show $r_G(\sigma(x)) = \mathrm{SQ}(\kappa(x))$ for all cohomology classes of all conjugation spaces. By Lemma 2.1 and Fact 2.2, it suffices to do so for $X = (\mathbf{CP}^\infty)^n$ (which is a conjugation space by [2, Proposition 4.5]) and x, the n-fold cross product of the generator v because $X^\tau = (\mathbf{RP}^\infty)^n$ and $\kappa(x) = w \times \cdots \times w$ in this case. For $n = 1$ the identity is true since we already know a_1. The general case reduces to the case $n = 1$ because of the multiplicativity of the maps κ, σ, r_G and SQ: writing $v_i \in H^2(X)$ for the pull-back of v induced by the projection $X \to \mathbf{CP}^\infty$ onto the i-th factor, we get

$$\begin{aligned} r_G\big(\sigma(x)\big) &= r_G\big(\sigma(v \times \cdots \times v)\big) = r_G\big(\sigma(v_1 \cdots v_n)\big) = r_G\big(\sigma(v_1)\big) \cdots r_G\big(\sigma(v_n)\big) \\ &= \mathrm{SQ}\big(\kappa(v_1)\big) \cdots \mathrm{SQ}\big(\kappa(v_n)\big) = \mathrm{SQ}\big(\kappa(v_1 \cdots v_n)\big) = \mathrm{SQ}\big(\kappa(x)\big). \qquad \square \end{aligned}$$

Proof of Corollary 1.2. We have for $x \in H^{2n}(X)$

$$r(x) = r\big(p(\sigma(x))\big) = p\big(r_G(\sigma(x))\big) = p\big(\mathrm{Sq}^n(\kappa(x))\big) = \kappa(x)^2. \qquad \square$$

Proof of Theorem 1.3. As in the proof of Theorem 1.1, it suffices to show the claimed identity for $X = (\mathbf{CP}^\infty)^n$ and $x = v \times \cdots \times v$. Again, the general case can be reduced to $n = 1$, where we find

$$\kappa\big(\mathrm{Sq}(v)\big) = \kappa\big(v + v^2\big) = \kappa(v) + \kappa(v)^2 = \mathrm{Sq}\big(\kappa(v)\big). \qquad \square$$

3. Remarks

Let X be a non-singular complex projective variety defined over the reals such that its real locus X^τ is non-empty. In what follows, all algebraic cycles in X are understood to be defined over the reals. Borel and Haefliger have shown that if $H_*(X)$ and $H_*(X^\tau)$ are generated by algebraic cycles, then the restriction λ of cycles in X to their real locus induces a degree-halving isomorphism $H_{2*}(X) \to H_*(X^\tau)$ respecting intersection products [1, §5.15]. They also show that if $H^*(X^\tau)$ is generated by algebraic cycles and $x \in H^*(X)$ is Poincaré dual to a linear combination of non-singular subvarieties, then the identity in Theorem 1.3 holds, and they ask whether it holds more generally [1, §5.17].

Krasnov has proved that for a variety X as above, Theorem 1.1 holds for cohomology classes Poincaré dual to algebraic cycles, where κ is the Poincaré transpose of λ and σ the canonical section [3, Theorem 4.2]. This implies that if $H_*(X)$ is generated by algebraic cycles, then so is $H_*(X^\tau)$ [4, Theorem 0.1]. Moreover, X is a conjugation spaces in the sense of [2].

In a topological framework van Hamel has recently shown that certain topological manifolds with involutions are conjugation spaces [6, Theorem]. The necessary assumptions are formulated in terms of topological cycles.

The following simple example shows that in general the existence of a degree-halving multiplicative isomorphism $\kappa : H^*(X) \to H^*(X^\tau)$ by itself does not imply that (X, τ) is a conjugation space.

Example 1. Let $X = S^2 \times S^4$ be equipped with the componentwise involution τ which is the identity for S^2 and for S^4 has fixed point set S^1. So $X^\tau = S^2 \times S^1$. Clearly there is a degree-halving multiplicative isomorphism $\kappa : H^*(X) \to H^*(X^\tau)$. It is easy to check there is also a multiplicative section $\sigma : H^*(X) \to H_G^*(X)$. But (X, τ) is not a conjugation space: the restriction map

$$r_G : H_G^*\bigl(S^2 \times S^4\bigr) \cong H^*\bigl(S^2 \times S^4\bigr) \otimes \mathbf{F}_2[u] \to H^*\bigl(S^2 \times S^1\bigr) \otimes \mathbf{F}_2[u]$$

is given by $s_2 \otimes 1 \mapsto s_2 \otimes 1$ and $s_4 \otimes 1 \mapsto s_1 \otimes u^3$, where $s_n \in H^n(S^n)$ denotes the generator. Hence the conjugation equation does not hold. Of course, $S^2 \times S^4$ with the different componentwise involution $\tilde{\tau}$ which has $S^1 \subset S^2$ and $S^2 \subset S^4$ as fixed point sets (and hence $X^{\tilde{\tau}} = S^1 \times S^2 \cong X^\tau$) is a conjugation space.

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