Available online at www.sciencedirect.com

SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 342 (2006) 165–168

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Algebra/Topology

# Stabilization of the Witt group

Max Karoubi

*Département de mathématiques, UMR 7586 du CNRS, case 7012, université Paris 7, 2, place Jussieu, 75251 Paris cedex 05, France*

Received 26 September 2005; accepted after revision 16 November 2005

Presented by Alain Connes

**Abstract**

In this Note, using an idea due to Thomason, we define a "homology theory" on the category of rings which satisfies excision, exactness, homotopy (in the algebraic sense) and periodicity of order 4. For regular noetherian rings, we find Balmer's higher Witt groups. For more general rings, this homology is isomorphic to the *KT*-theory of Hornbostel, inspired by the work of Williams. For real or complex $C^*$-algebras, we recover – up to 2 torsion – topological $K$-theory. ***To cite this article: M. Karoubi, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***


**Résumé**

**Stabilisation du groupe de Witt.** En utilisant une idée due à Thomason, nous définissons dans cette Note une « théorie de l'homologie » sur la catégorie des anneaux qui satisfait aux propriétés d'excision, d'exactitude, d'homotopie (au sens algébrique) et de périodicité d'ordre 4. Pour les anneaux noethériens réguliers, nous retrouvons les groupes de Witt supérieurs de Balmer. Pour des anneaux plus généraux, cette homologie est isomorphe à la *KT*-théorie définie par Hornbostel et inspirée par le travail de Williams. Pour les algèbres stellaires, réelles ou complexes, nous retrouvons – à la 2 torsion près – la $K$-théorie topologique. ***Pour citer cet article : M. Karoubi, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***


## 1.

Let $A$ be a ring with an anti-involution $a \mapsto \bar{a}$ and let $\varepsilon$ be an element of the center of $A$ such that $\varepsilon\bar{\varepsilon} = 1$. We assume also that 2 is invertible in the ring. There are now well known definitions of the higher hermitian $K$-group (denoted by ${}_\varepsilon L_n(A)$, as in [5]) and the higher Witt group ${}_\varepsilon W_n(A)$: this is the cokernel of the map induced by the hyperbolic functor

$$K_n(A) \longrightarrow {}_\varepsilon L_n(A)$$

where the $K_n(A)$ denote the Quillen $K$-groups (which are defined for all values of $n \in \mathbf{Z}$). One of the fundamental results of higher Witt theory is the periodicity isomorphism (where $\mathbf{Z}' = \mathbf{Z}[1/2]$, cf. [4])

$${}_\varepsilon W_n(A) \otimes \mathbf{Z}' \cong {}_{-\varepsilon} W_{n-2}(A) \otimes \mathbf{Z}'.$$

*E-mail address:* karoubi@math.jussieu.fr (M. Karoubi).

1631-073X/$ – see front matter 
doi:10.1016/j.crma.2005.12.014

It is induced by the cup-product with a genuine element $u_2 \in {}_{-1}L_{-2}(\mathbf{Z}')$. By analogy with algebraic topology, we shall call $u_2$ the Bott element in Witt theory. This element is explicitly described in the following way. We consider the $2\times 2$ matrix (with the involution defined by $\bar{z} = z^{-1}$ and $\bar{t} = t^{-1}$ and where we put $\lambda = \bar{\lambda} = 1/2$), defined in [4] p. 243. This $2\times 2$ matrix represents an element of ${}_{-1}L_0(\mathbf{Z}'[t, t^{-1}, z, z^{-1}])$ whose image in ${}_{-1}L_{-2}(\mathbf{Z}') \cong {}_{-1}W_{-2}(\mathbf{Z}') \cong \mathbf{Z} \oplus \mathbf{Z}/2$ is a free generator (cf. [5] for the details).

**2.**

The higher Witt groups ${}_\varepsilon W_n(A)$ have not all the nice formal properties one should expect. For instance, a cartesian square of rings with anti-involutions (where the vertical maps are surjective)

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ A_2 & \longrightarrow & A' \end{array}$$

does not induce, in general, a long Mayer–Vietoris exact sequence of Witt groups

$$\longrightarrow {}_\varepsilon W_{n+1}(A') \longrightarrow {}_\varepsilon W_n(A) \longrightarrow {}_\varepsilon W_n(A_1) \oplus {}_\varepsilon W_n(A_2) \longrightarrow {}_\varepsilon W_n(A') \longrightarrow .$$

Following an idea due to Thomason [8], one may overcome this difficulty by *stabilizing* the higher Witt groups. More precisely, we define a new theory ${}_\varepsilon\mathfrak{W}_n(A)$ as the limit of the inductive system

$${}_\varepsilon W_n(A) \longrightarrow {}_{-\varepsilon} W_{n-2}(A) \longrightarrow {}_\varepsilon W_{n-4}(A) \longrightarrow \cdots$$

where the arrows are induced by the cup-product with the Bott element $u_2$ mentioned above. As a matter of fact, the periodicity map ${}_\varepsilon W_n(A) \longrightarrow {}_{-\varepsilon} W_{n-2}(A)$ can be factored as

$${}_\varepsilon W_n(A) \longrightarrow {}_{-\varepsilon} L_{n-2}(A) \longrightarrow {}_{-\varepsilon} W_{n-2}(A).$$

Therefore ${}_\varepsilon\mathfrak{W}_n(A)$ is also the limit of the inductive system

$${}_\varepsilon L_n(A) \longrightarrow {}_{-\varepsilon} L_{n-2}(A) \longrightarrow {}_\varepsilon L_{n-4}(A) \longrightarrow \cdots.$$

**Theorem 2.1.** *This new theory ${}_\varepsilon\mathfrak{W}_n(A)$ satisfies the following properties*:

(a) Homotopy invariance. *The polynomial extension $A \to A[t]$, where $\bar{t} = t$, induces an isomorphism*

$${}_\varepsilon\mathfrak{W}_n(A) \cong {}_\varepsilon\mathfrak{W}_n\big(A[t]\big).$$

(b) Exactness and excision. *From a cartesian square as above* (*with $\psi$ surjective*), *one deduces an isomorphism of the associated relative groups*

$${}_\varepsilon\mathfrak{W}_n(\phi) \cong {}_\varepsilon\mathfrak{W}_n(\psi)$$

*and therefore a* Mayer–Vietoris *exact sequence for all $n \in \mathbf{Z}$*

$$\longrightarrow {}_\varepsilon\mathfrak{W}_{n+1}(A') \longrightarrow {}_\varepsilon\mathfrak{W}_n(A) \longrightarrow {}_\varepsilon\mathfrak{W}_n(A_1) \oplus {}_\varepsilon\mathfrak{W}_n(A_2) \longrightarrow {}_\varepsilon\mathfrak{W}_n(A') \longrightarrow .$$

(c) Periodicity. *The cup-product with the* Bott *element induces the isomorphisms*

$${}_\varepsilon\mathfrak{W}_n(A) \cong {}_{-\varepsilon}\mathfrak{W}_{n-2}(A) \cong {}_\varepsilon\mathfrak{W}_{n-4}(A).$$

(d) Normalization. *Let us assume now that $A$ is a regular noetherian ring. Then the natural map*

$${}_\varepsilon W_0(A) \longrightarrow {}_\varepsilon\mathfrak{W}_0(A)$$

*is an isomorphism and the group ${}_\varepsilon\mathfrak{W}_1(A)$ is isomorphic to the cokernel of the map defined in* [5].

$$k_0(A) \longrightarrow {}_\varepsilon W_1(A).$$

*Moreover, the groups ${}_1\mathfrak{W}_n(A)$ coincide with the higher* Witt *groups of Balmer* [1].

**Sketch of the proof.** Periodicity is imposed by the definition (as in Thomason's theory). Homotopy invariance is a consequence of the same property for the Witt groups. Since the $L_n$-groups satisfy the excision and exactness properties for $n<0$ (cf. [6] for instance), this is also true of the theory ${}_\varepsilon\mathfrak{W}_n$: as we have noticed before, ${}_\varepsilon\mathfrak{W}_n(A)$ is also the limit of the inductive system

$$ {}_\varepsilon L_n(A) \longrightarrow {}_{-\varepsilon}L_{n-2}(A) \longrightarrow {}_\varepsilon L_{n-4}(A) \longrightarrow . $$

The rest of the theorem follows immediately from [5], Theorem 4.3 and from next theorem. □

**Theorem 2.2.** *The homology ${}_\varepsilon\mathfrak{W}_*(A)$ is isomorphic to the KT∗-theory of Hornbostel* (*cf.* [3], *Section* 5).

**Proof.** This $KT$-theory is the direct limit of the system

$$ {}_\varepsilon L_n(A) \longrightarrow {}_\varepsilon L_n(U_A) \longrightarrow \cdots \longrightarrow {}_\varepsilon L_n\big(U_A^r\big) \longrightarrow $$

where $U_A$ is the ring defined in [5], p. 263 and $U_A^r$ the $r$-iteration of the "$U$-construction". All the arrows above are $L_*$-module maps as defined in [4] p. 233 and [5] p. 276. This implies that the homomorphism

$$ {}_\varepsilon L_n(A) \longrightarrow {}_\varepsilon L_n\big(U_A^r\big) $$

is the cup-product with a well defined element $w_r$ in ${}_1L_0(U_{\mathbf{Z}'}^r)$ (this is probably related to the question 6.6 raised by Hornbostel in his paper [3]). On the other hand, as a consequence of the fundamental theorem of hermitian $K$-theory (cf. [5] p. 264), we have an isomorphism of $L_*$-modules between ${}_\varepsilon L_n(U_A^r)$ and ${}_\varepsilon L_n(U_A^{r+4})$ (as noticed also by Williams [9]). Therefore, the previous direct limit is simply the limit of the system

$$ {}_\varepsilon L_n(A) \longrightarrow {}_\varepsilon L_{n-4}(A) \longrightarrow \cdots $$

where the arrows are defined by the cup-product with a specific element $w$ in ${}_1L_{-4}(\mathbf{Z}')$. On the other hand, we know that if we apply this construction to the ring $A=\mathbf{Z}'$ and $\varepsilon=1$, we find an isomorphism between ${}_1W_0(Z')$ and ${}_1L_{-4}(\mathbf{Z}')={}_1W_{-4}(\mathbf{Z}')$ (because the ring $\mathbf{Z}'$ is regular). As a matter of fact, we find a chain of isomorphisms

$$ {}_1W_0\big(\mathbf{Z}'\big) \cong {}_1L_0\big(U_{\mathbf{Z}'}^2\big) \cong {}_1L_0\big(U_{\mathbf{Z}'}^3\big) \cong {}_1L_0\big(U_{\mathbf{Z}'}^3\big) \cong {}_1L_0\big(U_{\mathbf{Z}'}^4\big) \cong {}_1L_{-4}\big(\mathbf{Z}'\big). $$

I claim that $w$, the image of 1 by this chain of isomorphisms, is $(u_2)^2$ up to a unipotent element. This is exactly the well known computation of the classical Witt ring of $\mathbf{Z}'$ which is $\mathbf{Z}\oplus\mathbf{Z}/2$ generated by the classes of the following elements in the Grothendieck Witt group: $\langle x^2\rangle$ and $\langle x^2\rangle - \langle 2x^2\rangle$. □

If $A$ is regular noetherian and $\varepsilon=1$, Hornbostel has proved moreover in [3] that $KT_n$(A) is isomorphic to the $n$-Witt group defined by Balmer, which proves the last part of Theorem 2.1.

**Remark 1.** For simplicity's sake, we have just considered hermitian $K$-theory groups. One could have taken as well homotopy colimits of the corresponding classifying spaces, using for instance the machinery developed in [4], Section 1, greatly generalized by Schlichting.

**Remark 2.** Using [2] and most of the above properties of the theory ${}_\varepsilon\mathfrak{W}_n$, Schlichting was able to prove cdh descent for the theory ${}_\varepsilon\mathfrak{W}_*$ extended to the category of (commutative) schemes of finite type over a field of characteristic 0 [7].

## 3.

When $A$ is a Banach algebra, we may consider the topological analogs of the previous definitions. In that case, the group ${}_\varepsilon\mathfrak{W}_n^{\mathrm{top}}(A)$ is simply isomorphic to ${}_\varepsilon W_n^{\mathrm{top}}(A)\otimes\mathbf{Z}'$. One should also notice that ${}_\varepsilon W_n^{\mathrm{top}}(A)\otimes\mathbf{Z}'$ is isomorphic to ${}_\varepsilon W_n(A)\otimes\mathbf{Z}'$ as a consequence of the periodicity theorem and the well-known computation of ${}_\varepsilon W_0$ and ${}_\varepsilon W_1$. Finally, if $A$ is a $C^*$algebra, it is also well known that ${}_1W_n^{\mathrm{top}}(A)$ is isomorphic to the topological $K$-theory of $A$.

## Acknowledgements

It is a pleasure to thank Jens Hornbostel and Marco Schlichting for their interesting remarks and suggestions after a first draft of this Note.

## References

[1] P. Balmer, Witt groups, in: Handbook of $K$-Theory, Springer-Verlag, 2005.
[2] G. Cortinas, C. Haesemeyer, M. Schlichting, C. Weibel, Cyclic homology, cdh-cohomology and negative $K$-theory, http://www.math.uiuc.edu/K-theory/0722/.
[3] J. Hornbostel, $A^1$ representability of Hermitian $K$-theory and Witt groups, Topology 44 (2005) 661–687.
[4] M. Karoubi, Théorie de Quillen et homologie du groupe orthogonal, Ann. Math. 112 (1980) 207–257.
[5] M. Karoubi, Le théorème fondamental de la $K$-théorie hermitienne, Ann. Math. 112 (1980) 259–282.
[6] M. Karoubi, O. Villamayor, $K$-théorie algébrique et $K$-théorie topologique II, Math. Scand. 32 (1973) 57–86.
[7] M. Schlichting, in press.
[8] R. Thomason, Algebraic $K$-theory and etale cohomology, Ann. Sci. École Norm. Sup. 13 (1985) 437–552.
[9] B. Williams, Quadratic $K$-theory and geometric topology, in: Handbook of $K$-Theory, Springer-Verlag, 2005.