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Dynamical Systems

Normal forms for nonlinear control systems with scalar output

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Abstract

We propose a normal form for nonlinear control systems with scalar output. We follow an approach proposed by Poincaré and adapted for control systems by Kang and Krener which consists of analyzing, step-by-step, the action of the change of coordinates on the system. ***To cite this article: I.A. Tall, M. Balde, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Formes normales pour les systèmes de contrôle non linéaires mono-sortie. Nous présentons dans cette Note une forme normale pour les sytèmes de contrôle non linéaires mono-sortie. Nous suivons une approche proposée par Poincaré et adaptée aux systèmes de contrôle par Kang et Krener, consistant à analyser, pas-à-pas, l'action du changement de coordonnées sur le système. ***Pour citer cet article : I.A. Tall, M. Balde, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Le problème consistant à ramener, par un changement de coordonnées, l'évolution de l'état d'un système dynamique non linéaire à celle d'un système linéaire fait l'objet de travaux qui remontent à Henri Poincaré [9]. Lorsqu'un tel changement de coordonnées existe on dit que le système est équivalent à sa partie linéaire ou linéarisable ; autrement le système est dit non linéarisable. Poincaré a montré [9] qu'un système dynamique est linéarisable autour d'un point d'équilibre si le spectre de sa partie linéaire est non résonnant. Pour les systèmes dont le spectre est résonant, il adopta une méthode basée sur l'application, pas-à-pas, d'un changement de coordonnées transformant ainsi le système dynamique en une forme normale ne contenant que des termes résonants.

Kang et Krener [6,4] ont adapté la méthode de Poincaré aux systèmes de contrôle ; ces derniers pouvant être considérés comme des systèmes dynamiques paramétrés. Il s'avère que leur idée a été très fructueuse et fût appliquée avec succés à la classification par bouclage des systèmes de contrôle, à l'étude de leurs bifurcations, à celle de leurs symétries, à leur stabilité, et à leur observabilité (voir [5,10,3,11]).

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Pour l'observabilité des systèmes de contrôle non linéaires, le premier objectif a été de ramener la question délicate de la construction d'observateurs non linéaires à celle beaucoup plus simple d'observateurs linéaires. Krener et Isidori [7] et Krener et Respondek [8] ont établi l'existence d'une classe de systèmes non linéaires pouvant être transformée, par changement de coordonnées, en une forme normale dont les termes non linéaires ne dépendent que du contrôle et de la sortie du système. Dans leurs travaux [7,8] la notion de linéarisation par injection de sortie a été introduite ce qui a permis la construction d'observateurs non linéaires dont l'erreur suit une dynamique linéaire. Cependant, la condition d'intégrabilité exigée dans [7] et [8] n'est pas générique, c'est-à-dire non satisfaite par la plupart des systèmes. D'ailleurs, même dans le cas où cette condition est satisfaite, le calcul explicite du changement de coordonnées nécessite la résolution, souvent difficile, d'équations aux dérivées partielles.

Dans cette Note nous donnons des formes normales formelles pour les systèmes de contrôle mono-sortie en analysant, pas-à-pas, l'action de la partie homogène d'une transformation formelle sur la partie homogène de même degré du système. Notons qu'une approache similaire a été utilisée pour les systèmes de contrôle mono-sortie à temps discrets [2]. La motivation d'introduire les formes normales est double : (i) permettre la construction d'observateurs non linéaires en utilisant de nouvelles techniques apparues dans ce domaine [5] ; (ii) trouver des conditions nécessaires et suffisantes pour l'observablité des systèmes de contrôle non linéaires mono-sortie. Ces questions seront abordées dans un article en préparation.

1. Introduction

The problem of transforming a nonlinear dynamical system into a linear system, via change of coordinates, goes back to Henri Poincaré's work in his PhD thesis [9]. If such a change of coordinates exist, the system is said to be linearizable; otherwise it is non linearizable. Poincaré showed (see also [1]) that a dynamical system is linearizable around an equilibrium point if the eigenvalues of its linearization are non resonant. In the case when the eigenvalues of the linearization are resonant, he performed a step-by-step change of coordinates that transforms the original system into a normal form, where only non resonant terms are present.

Kang and Krener [6,4] adapted Poincaré's classical technique to control systems, that could be viewed as parameterized dynamical systems. Their idea turned out to be very fruitful and has been applied successfully to study feedback classification, bifurcations, symmetries, stability, and observability of control systems (see [5,10,3,11] and the references therein).

For observability of control systems, the original goal was to reduce the difficult problem of constructing nonlinear observers to that simple of linear observers. Krener and Isidori [7] and Krener and Respondek [8] showed that there exists a class of nonlinear systems that can be transformed into a normal form where all nonlinearities depend only on the control and output of the system. In their work [7,8] the notion of linearization by output injection was introduced which allowed the construction of nonlinear observers with linearizable error dynamics. However, the necessary integrability condition demanded in [7] and [8] is not generic which means not satisfied by many systems. And, even when the integrability condition is satisfied, finding the change of coordinates requires the difficult task of solving partial differential equations.

In this Note we study normal forms for nonlinear control systems with scalar output. We will analyze, step-by-step, the action of homogeneous change of coordinates on the corresponding part of the same degree of the system. Notice that similar approach has been used for discrete-time systems with scalar output [2] for quadratic terms. The main purpose of studying normal forms is two-fold: (1) construct nonlinear observers using new techniques (see [5]); (2) find necessary and sufficient conditions for observability of nonlinear control systems with scalar output. Those questions will be addressed in an upcoming paper.

2. Normal forms

2.1. Linearly observable case

Consider the class of systems $\dot{x} = f(x, v)$, $y = Cx$, where $x(\cdot) \in \mathbb{R}^n$ is the state of the system, $v(\cdot) \in \mathbb{R}$ is the input or the control parameter and $y(\cdot) \in \mathbb{R}$ is the output.

We suppose that $f(0, 0) = 0$ and assume the linear approximation of the system, given by

$$\dot{x} = F_0 x + Bv, \quad F_0 = \frac{\partial f}{\partial x}(0,0), \ B = \frac{\partial f}{\partial v}(0,0), \qquad y = Cx$$

to be observable, that is, rank$[C^T, \ (CF_0)^T, \ldots, (CF_0^{n-1})^T] = n$. Using a linear change of coordinates, we can write the Taylor expansion of the system as follows

$$\dot{x} = Ax + Gy + Bv + \sum_{m=2}^{d} f^{[m]}(x,v) + O^{d+1}(x,v), \qquad y = Cx = x_1, \tag{1}$$

where $(A + GC, C)$ is in the dual Brunovský canonical observer form, G and B are $n \times 1$ matrices. Above and throughout this section, the subscript $[m]$ will denote homogeneous terms of degree m in the indicated variables, and $O^{d+1}(x,v)$ stands for terms of degree $d+1$ and higher. Define the subset $W_1 = \{(x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \colon x_2 = \cdots = x_n = 0\}$. We prove the following result.

Theorem 2.1. *Given $d \in \mathbb{N}$, there is a degree d polynomial change of coordinates*

$$\bar{x} = \Phi(x) = x + \sum_{m=2}^{d} \varphi^{[m]}(x),$$

that transforms the system (1) *into*

$$\dot{\bar{x}} = A\bar{x} + Bv + Gy + \sum_{m=2}^{d} \big(\beta^{[m]}(y,v) + \bar{f}^{[m]}(\bar{x},v)\big) + O^{d+1}(\bar{x},v), \qquad y = C\bar{x} = \bar{x}_1,$$

where $\beta(y,v) = \sum_{m=2}^{d} \beta^{[m]}(y,v)$ is a degree d polynomial depending only on y and v, and

$$\bar{f}^{[m]}(\bar{x},v) = \sum_{k=0}^{m-1} v^k \bar{F}^{[m-k]}(\bar{x}) \tag{2}$$

is such that for any $1 \leqslant j \leqslant n-1$, any $1 \leqslant s \leqslant n$, and any $1 \leqslant k \leqslant m-1$,

$$\text{(i) } \bar{F}_j^{[m]}(\bar{x}) = 0, \quad \text{(ii) } \frac{\partial^m \bar{F}_n^{[m]}}{\partial \bar{x}_1^{m-1} \partial \bar{x}_s} = 0, \quad \textit{and} \quad \text{(iii) } \frac{\partial^{m-k} \bar{F}^{[m-k]}}{\partial x_1^{m-k}} = 0. \tag{3}$$

Proof. Suppose that for some $2 \leqslant m \leqslant d$ the system has been brought to the form

$$\dot{x} = Ax + Bv + Gy + \sum_{q=2}^{m-1} \big(\beta^{[q]}(y,v) + \bar{f}^{[q]}(x,v)\big) + f^{[m]}(x,v) + O^{m+1}(x,v), \qquad y = Cx = x_1,$$

by a polynomial change of coordinates of degree $m-1$, where $\bar{f}^{[q]}(\bar{x},v)$, $2 \leqslant q \leqslant m-1$, is in the form (2), (3). Then we seek for a coordinates change $\bar{x} = x + \varphi^{[m]}(x)$ to take the system into

$$\dot{\bar{x}} = A\bar{x} + Bv + Gy + \sum_{q=2}^{m} \big(\beta^{[q]}(y,v) + \bar{f}^{[q]}(\bar{x},v)\big) + O^{m+1}(\bar{x},v), \qquad y = C\bar{x} = \bar{x}_1,$$

with $\bar{f}^{[m]}(\bar{x},v)$ given by (2), (3). Notice that the terms $\bar{f}^{[q]}(\bar{x},v)$ and $\beta^{[q]}(y,v)$ for $2 \leqslant q \leqslant m-1$ remain invariant. Such polynomials should satisfy the homological equation

$$-A\varphi^{[m]}(x) + \frac{\partial \varphi_n^{[m]}}{\partial x}(F_0 x + Bv) = \beta^{[m]}(x,v) + \bar{f}^{[m]}(x,v) - f^{[m]}(x,v) \tag{4}$$

with the constraint that $\varphi_1^{[m]}(x) = 0$. Decompose $f^{[m]}(x,v)$, $\bar{f}^{[m]}(x,v)$, and $\beta^{[m]}(y,v)$ as

$$f^{[m]}(x,v) = \sum_{k=0}^{m} v^k F^{[m-k]}(x), \qquad \bar{f}^{[m]}(x,v) = \sum_{k=0}^{m} v^k \bar{F}^{[m-k]}(x), \qquad \beta^{[m]}(y,v) = \sum_{k=0}^{m} v^k \gamma^{[m-k]}(y).$$

Thus, the homological equation (4) splits into the following three subsystems of equations

$$\gamma^{[m]}(x_1)+\bar F^{[m]}(x)-F^{[m]}(x)=-A\varphi^{[m]}(x)+\frac{\partial\varphi_n^{[m]}}{\partial x}F_0x, \tag{5}$$

$$\gamma^{[m-1]}(x_1)+\bar F^{[m-1]}(x)-F^{[m-1]}(x)=\frac{\partial\varphi_n^{[m]}}{\partial x}B, \tag{6}$$

$$\gamma^{[m-k]}(x_1)+\bar F^{[m-k]}(x)-F^{[m-k]}(x)=0,\quad \text{for all } 2\leqslant k\leqslant m. \tag{7}$$

Of course if $\varphi^{[m]}(x)$ is completely determined from Eq. (5), then $\gamma^{[m-1]}(x_1)$ will allow to cancel terms of the form x_1^{m-1} from $\bar F^{[m-1]}(x)$, and with $\gamma^{[m-k]}(x_1)$ we will cancel terms of the from x_1^{m-k} from $\bar F^{[m-k]}(x)$. This implies that the vector field $\bar F^{[m-k]}(x)$ will satisfy (3).

Now let us consider (5) and let us put $\bar F_1^{[m]}(x)=\cdots=\bar F_{n-1}^{[m]}(x)=0$. We then define the vector $\varphi^{[m]}(x)$ recursively as following: Set $\varphi_1^{[m]}(x)=0$ and for any $1\leqslant j\leqslant n-1$ take $\varphi_{j+1}^{[m]}(x)=L_{F_0x}\varphi_j^{[m]}(x)-\gamma_j^{[m]}(x_1)+F_j^{[m]}(x)$. It follows that the subsystem (5) reduces to

$$\frac{\partial\varphi_n^{[m]}}{\partial x}F_0x=L_{F_0x}\varphi_n^{[m]}(x)=\gamma_n^{[m]}(x_1)+\bar F_n^{[m]}(x)-F_n^{[m]}(x). \tag{8}$$

It is easy to show by iteration from (5) that $\varphi_n^{[m]}(x)=-\sum_{k=1}^{n-1}L_{F_0x}^{n-k-1}\gamma_k^{[m]}(x_1)+\sum_{k=1}^{n-1}L_{F_0x}^{n-k-1}F_k^{[m]}(x)$. Plugging this expression into (8) we deduce that

$$\begin{aligned}\bar F_n^{[m]}(x)&=-\sum_{k=1}^{n-1}L_{F_0x}^{n-k}\gamma_k^{[m]}(x_1)+\sum_{k=1}^{n-1}L_{F_0x}^{n-k}F_k^{[m]}(x)-\gamma_n^{[m]}(x_1)+F_n^{[m]}(x)\\&=-\sum_{k=1}^{n}L_{F_0x}^{n-k}\gamma_k^{[m]}(x_1)+\sum_{k=1}^{n}L_{F_0x}^{n-k}F_k^{[m]}(x).\end{aligned} \tag{9}$$

Let us remark, since $\gamma_k^{[m]}(x_1)=\beta_{1,k}x_1^m$ for all k, that $L_{F_0x}^{n-k}\gamma_k^{[m]}(x_1)=m\beta_{1,k}x_1^{m-1}x_{n-k+1}+\theta_k^{[m]}(x_1,\ldots,x_{n-k})$, where $\theta_k^{[m]}(x_1,\ldots,x_{n-k})$ are appropriate functions of the variables $x_1,\ldots,x_{n-k}$ whose coefficients depend on $\beta_{1,k}$. We will choose iteratively the coefficients $\beta_{1,k}$ so that

$$\begin{aligned}m\beta_{1,1}x_1^{m-1}&=\left[\frac{\partial}{\partial x_n}\left(\sum_{k=1}^{n}L_{F_0x}^{n-k}F_k^{[m]}(x)\right)\right]\Bigg|_{W_1},\\ m\beta_{1,2}x_1^{m-1}&=\left[\frac{\partial}{\partial x_{n-1}}\left(-L_{F_0x}^{n-1}\gamma_1^{[m]}(x_1)+\sum_{k=1}^{n}L_{F_0x}^{n-k}F_k^{[m]}(x)\right)\right]\Bigg|_{W_1},\\ &\ \vdots\\ m\beta_{1,n}x_1^{m-1}&=\left[\frac{\partial}{\partial x_1}\left(-\sum_{k=1}^{n-1}L_{F_0x}^{n-k}\gamma_k^{[m]}(x_1)+\sum_{k=1}^{n}L_{F_0x}^{n-k}F_k^{[m]}(x)\right)\right]\Bigg|_{W_1}.\end{aligned}$$

Now, to see that $\bar F_n^{[m]}$ satisfies (3)-(i) it suffices to differentiate w.r.t. x_{n-s+1} for $1\leqslant s\leqslant n$.

$$\begin{aligned}\frac{\partial\bar F_n^{[m]}}{\partial x_{n-s+1}}&=\frac{\partial}{\partial x_{n-s+1}}\left(-\sum_{k=1}^{n}L_{F_0x}^{n-k}\gamma_k^{[m]}(x_1)+\sum_{k=1}^{n}L_{F_0x}^{n-k}F_k^{[m]}(x)\right)\\&=\frac{\partial}{\partial x_{n-s+1}}\left(-\sum_{k=1}^{s}L_{F_0x}^{n-k}\gamma_k^{[m]}(x_1)+\sum_{k=1}^{n}L_{F_0x}^{n-k}F_k^{[m]}(x)\right)\end{aligned}$$

because for $k\geqslant s+1$, $L_{F_0x}^{n-k}\gamma_k^{[m]}(x_1)$ depends only on the variables $x_1,\ldots,x_{n-s}$. Since

$$\frac{\partial}{\partial x_{n-s+1}}\left(-\sum_{k=1}^{s} L_{F_0 x}^{n-k}\gamma_k^{[m]}(x_1)+\sum_{k=1}^{n} L_{F_0 x}^{n-k}F_k^{[m]}(x)\right)$$
$$=-m\beta_{1,s}x_1^{m-1}+\frac{\partial}{\partial x_{n-s+1}}\left(-\sum_{k=1}^{s-1} L_{F_0 x}^{n-k}\gamma_k^{[m]}(x_1)+\sum_{k=1}^{n} L_{F_0 x}^{n-k}F_k^{[m]}(x)\right)$$

it follows that

$$\left.\frac{\partial \bar F_n^{[m]}}{\partial x_{n-s+1}}\right|_{W_1}=-m\beta_{1,s}x_1^{m-1}+\left.\left[\frac{\partial}{\partial x_{n-s+1}}\left(-\sum_{k=1}^{s-1} L_{F_0 x}^{n-k}\gamma_k^{[m]}(x_1)+\sum_{k=1}^{n} L_{F_0 x}^{n-k}F_k^{[m]}(x)\right)\right]\right|_{W_1}=0.$$

Therefore $\bar F_n^{[m]}(x)$ does not contain terms of the form $x_1^{m-1}x_{n-s+1}$ for $1\leqslant s\leqslant n$ which implies that $\bar F_n^{[m]}(\bar x)$ does not contain terms of the form $\bar x_1^{m-1}\bar x_s$, $1\leqslant s\leqslant n$. This achieves the proof. □

2.2. Linearly non-observable case

Now, suppose that the linearization of the system is non-observable, that is it contains an r-dimensional linearly non-observable part and its Taylor expansion is

$$\begin{aligned}
\dot x &= Ax+Gy+Bv+\sum_{m=2}^{d} f^{[m]}(x,z,v)+O^{d+1}(x,z,v),\\
\dot z &= Jz+\sum_{m=2}^{d} g^{[m]}(x,z,v)+O^{d+1}(x,z,v),\\
y &= Cx=x_1,\quad (x,z)^T\in\mathbb{R}^{n+r},\quad v\in\mathbb{R},
\end{aligned}\tag{10}$$

where $(A+GC,C)$ is again in the dual Brunovský canonical observer form and J a diagonal matrix. In this section $f^{[m]}(x,z,v)$ and $g^{[m]}(x,z,v)$ denote homogeneous vector fields of degree m in the indicated variables.

If we assume that the spectrum $\sigma(J)=\{\lambda_1,\ldots,\lambda_r\}$ of J is *non-resonant*, that is, no relations of the form $\lambda_{i_1}+\cdots+\lambda_{i_m}-\lambda_j=0$, then we have the following generalization of Theorem 2.1.

Theorem 2.2. *Given $d\in\mathbb{N}$, there is a degree d polynomial change of coordinates*

$$\bar x=\Phi(x,z)=x+\sum_{m=2}^{d}\varphi^{[m]}(x,z),\qquad \bar z=\Psi(x,z)=z+\sum_{m=2}^{d}\psi^{[m]}(x,z)$$

that transforms the system (10) *into*

$$\begin{aligned}
\dot{\bar x} &= A\bar x+Bv+Gy+\sum_{m=2}^{d}\bigl(\beta^{[m]}(y,v)+\bar f^{[m]}(\bar x,\bar z,v)\bigr)+O^{d+1}(\bar x,\bar z,v),\\
\dot{\bar z} &= J\bar z+\sum_{m=2}^{d}\bigl(\alpha^{[m]}(y,v)+\bar g^{[m]}(\bar x,\bar z,v)\bigr)+O^{d+1}(\bar x,\bar z,v),\\
y &= C\bar x=\bar x_1,
\end{aligned}$$

where $\alpha(y,v)=\sum_{m=2}^{d}\alpha^{[m]}(y,v)$ and $\beta(y,v)=\sum_{m=2}^{d}\beta^{[m]}(y,v)$ are respectively, degree d vector fields and degree d polynomials depending only on y and v.

The vector fields $\bar f^{[m]}(\bar x,\bar z,v)=\sum_{k=0}^{m}v^k\bar F^{[m-k]}(\bar x,\bar z)$ and $\bar g^{[m]}(\bar x,\bar z,v)=\sum_{k=1}^{m}v^k\bar G^{[m-k]}(\bar x,\bar z)$ are such that for any $1\leqslant j\leqslant n-1$, for any $1\leqslant s\leqslant n$ and for any $1\leqslant k\leqslant m$ we have

$$\text{(i)}\ \bar F_j^{[m]}(\bar x,\bar z)=0,\quad \text{(ii)}\ \frac{\partial^m\bar F_n^{[m]}}{\partial\bar x_1^{m-1}\partial\bar x_s}=0,\quad \text{(iii)}\ \frac{\partial^{m-1}\bar G^{[m-1]}}{\partial\bar x_1^{m-2}\partial\bar x_2}=0,$$
$$\text{(iv)}\ \frac{\partial^{m-k}\bar F^{[m-k]}}{\partial\bar x_1^{m-k}}=0,\quad \text{(v)}\ \frac{\partial^{m-k}\bar G^{[m-k]}}{\partial\bar x_1^{m-k}}=0.$$

The proof of Theorem 2.2 follows the same line as that of Theorem 3 but is a little tricker. Remark that when $v = 0$, that is, the system is dynamical, then $\bar{g}^{[m]}(\bar{x}, \bar{z}, v) = 0$, which is coherent with Poincaré's result.

The co-dimension of the normal form is (because $\bar{G}^{[m]} = 0$)

$$\mathcal{C} = (n-1)\binom{n+r+m-1}{m} + n + r + nm + rm + r\binom{n+r+m-1}{m}$$
$$= (n+r-1)\binom{n+r+m-1}{m} + (n+r)\binom{2+m-1}{m}$$

which is $(n+r-1)\binom{n+r+m-1}{m} \triangleq$ dimension of coordinates change $(\varphi^{[m]}(x,z), \psi^{[m]}(x,z))$ (because $\varphi_1^{[m]} = 0$) plus $(n+r)\binom{2+m-1}{m} \triangleq$ dimension of $(\beta^{[m]}(y,v), \alpha^{[m]}(y,v))$. This justifies the unicity of the normal form.

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