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C. R. Acad. Sci. Paris, Ser. I 341 (2005) 519–523

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Numerical Analysis

Weights computation for simplicial Whitney forms of degree one

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Received 31 January 2005; accepted after revision 1 September 2005

Presented by Philippe G. Ciarlet

Abstract

We investigate some simple techniques of computation of the weights ('moments') of simplicial Whitney p-forms of first polynomial degree. The classical metric-dependent computation of weights is shown to be equivalent to an affine one, more suitable in the context of differential forms. ***To cite this article: F. Rapetti, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Calcul des poids pour les formes de Whitney simpliciales de degré polynomial un. On étudie quelques techniques simples de calcul des coefficients ('moments') des p-formes de Whitney de degré un sur les p-simplexes. On montre l'équivalence entre la méthode classique, de type métrique, pour le calcul de ces poids, et une méthode de type affine, mieux adaptée au contexte des formes différentielles. ***Pour citer cet article : F. Rapetti, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Les éléments de Whitney [1,5,6] sont peut-être les plus utilisés pour approcher des champs scalaires ou vectoriels en électromagnétisme. Dans cette Note, on reprend, dans la Section 2.1, une idée de Alain Bossavit parue dans [2, §1.7.1, §1.7.2] sur une façon de définir les formes de Whitney de degré un. En considérant ces formes comme un outil pour décrire une ligne (ou une surface, etc.) par des sommes pondérées (ou « chaînes ») d'arêtes (ou de faces, etc.) d'un maillage donné $\mathtt{m}$ sur le domaine $\Omega \subset \mathbb{R}^d$ considéré, on arrive à la Définition 2.1. Les coefficients de ces sommes sont les *poids* de la ligne (ou de la surface) dans la chaîne et la manière de les attribuer est le point central dans la construction des formes de Whitney. Au §2.2, cette idée conduit à des stratégies simples et équivalentes de calcul des poids des p-formes, $0 \leqslant p \leqslant d$, sur un simplexe quelconque de même dimension p. La Proposition 2.2 présente la méthode classique : on peut calculer les intégrales qui définissent les poids par la formule de quadrature du point milieu. La méthode classique répose sur la définition d'une métrique sur l'espace affine ambiante, pour calculer des quantités, les poids, qui sont indépendants de toute métrique. Par contre, la Proposition 2.3 et le Corollaire 2.4 dérivent de la Définition 2.1 : ils montrent comment ces poids sont calculables de façon affine, à partir seulement de la connaissance des coordonnées barycentriques et en fournissent une interprétation géométrique. Un outil de type

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.09.005

affine est mieux adapté dans le contexte des formes différentielles. Parmi les applications possibles, il permet de traiter aisément le problème du calcul des poids pour les formes de Whitney d'ordre supérieur, comme on le montrera dans un travail futur.

1. Introduction and notations

Whitney elements [1,5,6] are perhaps the most widely used finite elements in computational electromagnetics. In this article, we present in Section 2.1 the idea of Alain Bossavit described in [2, §1.7.1, §1.7.2] about a way to define Whitney forms of polynomial degree one, and we develop it in Section 2.2 to design some simple strategies to compute their 'weights' on simplices. These strategies, the one classical metric-dependent and the other affine, are shown to be equivalent. A geometrical interpretation of the weights is also provided and we end, in Section 3, by some considerations.

Let d be the ambient dimension. Given a domain $\Omega \subset \mathbb{R}^d$, a simplicial mesh $\mathtt{m}$ in Ω is a tessellation of $\overline{\Omega}$ by d-simplices, under the condition that any two of them may intersect along a common facet, i.e., a common subsimplex of dimension $0 \leqslant p \leqslant (d-1)$. In dimension $d = 3$, which we shall assume when giving examples, this means along a common face, edge or node, but in no other way. Labels n, e, f, v are used for nodes, edges, etc., each with its own orientation, and w^n, w^e, etc., refer to the corresponding Whitney forms. Boldface connotes 'discrete' objects, especially arrays of scalars. The sets of nodes, edges, faces, volumes (i.e., tetrahedra), are denoted by $\mathcal{N}_{\mathtt{m}}$, $\mathcal{E}_{\mathtt{m}}$, $\mathcal{F}_{\mathtt{m}}$, $\mathcal{V}_{\mathtt{m}}$. When in need for the generic symbol, we denote by $\mathcal{S}^p_{\mathtt{m}}$ the set of p-simplices of $\mathtt{m}$. The sets of p-simplices are linked by the incidence matrices $\mathbf{G}$, $\mathbf{R}$, $\mathbf{D}$, in dimension $d = 3$, otherwise the generic notation $\mathbf{d}$ will be used, suffixed by p if needed. For instance, $\partial f = \sum_{e \in \mathcal{E}_{\mathtt{m}}} \mathbf{R}^e_f e$ expresses the boundary of facet f as a formal linear combination of edges (such a thing is called a p-chain, with $p = 1$ here, see more details also in [1]). Symbol $\boldsymbol{\partial}$ will serve for $\mathbf{d}^t$, i.e., as a generic notation for the transposed $\mathbf{D}^t$, $\mathbf{R}^t$, $\mathbf{G}^t$. Recall that $\boldsymbol{\partial}$ is the boundary map for chains: e.g., given $\mathbf{c} = \{c^f\colon\ f \in \mathcal{F}_{\mathtt{m}}\}$, we have $\partial(\sum_{f \in \mathcal{F}_{\mathtt{m}}} c^f f) = \sum_{e \in \mathcal{E}_{\mathtt{m}}} (\boldsymbol{\partial}\mathbf{c})^e e$, with $\boldsymbol{\partial} = \mathbf{R}^t$ in this case. Our results hold for any spatial dimension d and all simplicial dimensions $0 \leqslant p \leqslant d$, but are stated as if d was 3. So we shall assume a specific p in proofs and prefer $\mathbf{R}$, $\mathbf{D}$, or $\mathbf{D}^t$, $\mathbf{R}^t$, to $\mathbf{d}$ or $\boldsymbol{\partial}$, but it should be clear each time that the proof has general validity.

2. Whitney forms

Fields, in electromagnetism, are observed via quantities, such as electro-motive forces, intensities, etc., which correspond to line integrals (*circulations*), surface integrals (*fluxes*), etc. A field (say, for example, b) then maps a p-manifold S ($p = 0$ for points, 1 for lines, and so on, and 2 in our example where S is a surface) to a number, here $\int_S b$. If w^f are facet elements, then b is represented by $\sum_{f \in \mathcal{F}_m} b_f w^f$ which we shall denote by $p_{\mathtt{m}} b$, being $p_{\mathtt{m}}$ the interpolation operator of a field on the Whitney forms. Suppose that we replace S by a p-chain $p^t_{\mathtt{m}} S = \sum_{f \in \mathcal{F}_{\mathtt{m}}} c^f f$, being $p^t_{\mathtt{m}}$ the operator mapping a surface in its 'finite' representation, and let us interpret the scalars b_f as the elementary values $\int_f b$ (fluxes, here). Then a natural approximation of $\int_S b$ is obtained by substituting $p^t_{\mathtt{m}} S$ for S. Hence an approximate knowledge of the field b, i.e., of all its measurable attributes, from the array $\mathbf{b} = \{b_f\colon\ f \in \mathcal{F}_{\mathtt{m}}\}$. The problem is then: "how best to represent S by a chain?". Solving it yields a *definition* of Whitney forms [8]: w^f, for instance, is, like the field b itself, a map from surfaces S to real numbers c^f, whose value we denote by $\int_S w^f$ or by $\langle w^f, S\rangle$. Note that, with this convention, $\langle b, p^t_{\mathtt{m}} S\rangle = \langle b, \sum_{f \in \mathcal{F}_{\mathtt{m}}} (\int_S w^f) f\rangle = \sum_{f \in \mathcal{F}_{\mathtt{m}}} \int_S w^f \langle b, f\rangle \equiv \langle p_{\mathtt{m}} b, S\rangle$. So, w^f is the Whitney form of polynomial degree one associated to f and the weight (or *moment*) of S in the chain $p^t_{\mathtt{m}} S$ is $\int_S w^f \equiv \langle w^f, S\rangle$. Note how this justifies the "$p^t_{\mathtt{m}}$" notation.

2.1. A generative formula for Whitney forms

We wish to represent a p-manifold by a p-chain. For $p = 0$, any point x of the meshed domain with position $\mathbf{x}$ can be represented as $x = \sum_{n \in \mathcal{N}_{\mathtt{m}}} \lambda_n(\mathbf{x}) n$, where $\lambda_n(\mathbf{x})$ is the barycentric coordinate of point x with respect to node n. Note that $\lambda_n(\mathbf{x}) \neq 0$ when the point x belongs to one of the tetrahedra with a vertex in n. So, the weight of x with respect to node n is $\lambda_n(\mathbf{x})$ and the 0-chain $p^t_{\mathtt{m}} x = \sum_{n \in \mathcal{N}_{\mathtt{m}}} \langle w^n, x\rangle n$ is its representation. The Whitney 0-form w^n is then λ_n, the hat function of the finite element method. The definition of $p_{\mathtt{m}} q = \sum_{n \in \mathcal{N}_{\mathtt{m}}} q_n w^n$ for a field q is obtained by transposition: $\langle q, p^t_{\mathtt{m}} x\rangle = \langle q, \sum_{n \in \mathcal{N}_{\mathtt{m}}} \lambda_n(\mathbf{x}) n\rangle = \sum_{n \in \mathcal{N}_{\mathtt{m}}} \lambda_n(\mathbf{x}) \langle q, n\rangle = \sum_{n \in \mathcal{N}_{\mathtt{m}}} q_n w^n(\mathbf{x}) = \langle \sum_{n \in \mathcal{N}_{\mathtt{m}}} q_n w^n, x\rangle \equiv \langle p_{\mathtt{m}} q, x\rangle$.

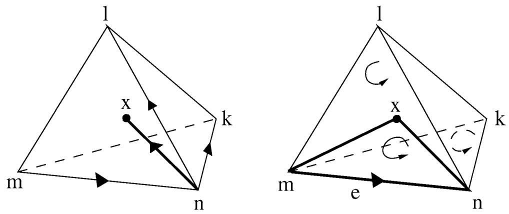

Fig. 1. Left: the 1-chain associated with the segment $x \vee n$ is $-\lambda_m(\mathbf{x})mn + \lambda_k(\mathbf{x})nk + \lambda_l(\mathbf{x})nl$. The minus sign in front of $\lambda_m(\mathbf{x})$ is due to the fact that the oriented edge mn starts in m, and ends in n. In terms of incidence numbers, $\mathbf{G}_m^{mn} = -1$ and $\mathbf{G}_n^{mn} = 1$. Edges e of the volume $\{m, k, n, l\}$ which do not have n as vertex make no contribution to the 1-chain. Right: the 2-chain associated with the surface $x \vee e$ is $\lambda_l(\mathbf{x})mnl + \lambda_k(\mathbf{x})kmn$. The orientation of edge e agrees with that induced on it by the two oriented faces mnl and kmn, so $\mathbf{R}_e^{mnl} = 1$ and $\mathbf{R}_e^{kmn} = 1$.

Hat functions have a double feature: they are the weights that represent a generic point as a linear combination of the mesh node positions, as well as the interpolants that allow for defining scalar functions from their nodal values at the mesh nodes. In the following, when $e = \{m, n\}$ and $f = \{l, m, n\}$, we denote the node l by $f - e$. Thus λ_{f-e} refers, in that case, to λ_l.

For $p = 1$, let xy be the oriented segment going from point x to point y. We know that $p_{\mathtt{m}}^t x = \sum_{n\in\mathcal{N}_{\mathtt{m}}} \langle w^n, x\rangle n$, and we figure out $p_{\mathtt{m}}^t xy$ by linearity: $p_{\mathtt{m}}^t xy = \sum_{n\in\mathcal{N}_{\mathtt{m}}} \langle w^n, y\rangle p_{\mathtt{m}}^t(x \vee n)$. As suggested in Fig. 1, $p_{\mathtt{m}}^t(x \vee n) = \sum_{e\in\mathcal{E}_{\mathtt{m}}} \mathbf{G}_n^e \lambda_{e-n}(\mathbf{x})e$. Thus $p_{\mathtt{m}}^t xy = \sum_{n\in\mathcal{N}_{\mathtt{m}}, e\in\mathcal{E}_{\mathtt{m}}} \mathbf{G}_n^e \lambda_{e-n}(\mathbf{x})\langle w^n, y\rangle e \equiv \sum_{e\in\mathcal{E}_{\mathtt{m}}} \langle w^e, xy\rangle e$. Hence

$$\langle w^e, xy\rangle = \sum_{n\in\mathcal{N}_{\mathtt{m}}} \mathbf{G}_n^e \lambda_{e-n}(\mathbf{x})\langle w^n, y\rangle.$$

Having $0 = \langle w^e, xx\rangle = \sum_{n\in\mathcal{N}_{\mathtt{m}}} \mathbf{G}_n^e \lambda_{e-n}(\mathbf{x})\langle w^n, x\rangle$ and d the dual (in the sense of [9]) of ∂, we get that

$$\langle w^e, xy\rangle = \sum_{n\in\mathcal{N}_{\mathtt{m}}} \mathbf{G}_n^e \lambda_{e-n}(\mathbf{x})\langle w^n, y - x\rangle = \sum_{n\in\mathcal{N}_{\mathtt{m}}} \mathbf{G}_n^e \lambda_{e-n}(\mathbf{x})\big\langle w^n, \partial(xy)\big\rangle = \sum_{n\in\mathcal{N}_{\mathtt{m}}} \mathbf{G}_n^e \lambda_{e-n}(\mathbf{x})\langle \mathrm{d}w^n, xy\rangle$$

for any 'small edge' xy, i.e., a segment xy entirely contained in the cluster of tetrahedra around e, and then $w^e = \sum_{n\in\mathcal{N}_{\mathtt{m}}} \mathbf{G}_n^e \lambda_{e-n}\, \mathrm{d}w^n$. The same mathematical steps can be done for $p = 2$ and $p = 3$, with obvious generalization when $d > 3$, yielding the following recursive definition of Whitney forms.

Definition 2.1. The differential $(p + 1)$-form w^σ given by

$$w^\sigma = \sum_{s\in\mathcal{S}_{\mathtt{m}}^p} \mathbf{d}_\sigma^s\, \lambda_{\sigma-s}\, \mathrm{d}w^s \tag{1}$$

is the Whitney form[1] of polynomial degree one associated to a $(p + 1)$-simplex σ, $0 \leqslant p \leqslant d - 1$.

2.2. Practical ways to compute weights for Whitney forms

Once a metric is introduced in the ambient affine space, differential forms such as w^n, w^e, etc., are in correspondence with scalar and vector fields (called 'proxy fields' – metric dependent, of coarse) whose expression is given in formula (1) by replacing d with **grad**. For instance, the vector $w^e = \lambda_\ell\, \mathbf{grad}\, \lambda_m - \lambda_m\, \mathbf{grad}\, \lambda_\ell$ is the vector field associated to the edge $e = \{\ell, m\}$. Its weight with respect to e (or *circulation* along e) is 1 and 0 on other simplices in $\mathtt{m}$ of matching dimension. Moreover, given two adjacent tetrahedra v and v' sharing a face f with e as part of ∂f, the tangential component of w^e is continuous across f. Thanks to this property, the set $W^1 = \mathrm{span}\{w^e\colon e \in \mathcal{E}_{\mathtt{m}}\}$ plays the role of internal Galerkin approximation space for the Sobolev space $H(\mathbf{curl}, \Omega)$ (see [4] for the definition and properties of $H(\mathbf{curl}, \Omega)$). Therefore, a vector field $h \in H(\mathbf{curl}, \Omega)$ can be represented in W^1 by $p_{\mathtt{m}} h = \sum_{e\in\mathcal{E}_{\mathtt{m}}} h_e w^e$ where the scalar h_e is the circulation of h along the mesh edge $e \in \mathcal{E}_{\mathtt{m}}$, i.e., the weight $\langle p_{\mathtt{m}} h, e\rangle$.

[1] For a node n, an edge e, a facet f and a tetrahedron v, formula (1) yields the following scalar or vector functions:

$$w^n = \lambda_n, \quad w^e = \sum_{n\in\mathcal{N}_{\mathtt{m}}} \mathbf{G}_e^n \lambda_{e-n}\, \mathrm{d}w^n, \quad w^f = \sum_{e\in\mathcal{E}_{\mathtt{m}}} \mathbf{R}_f^e \lambda_{f-e}\, \mathrm{d}w^e, \quad w^v = \sum_{f\in\mathcal{F}_{\mathtt{m}}} \mathbf{D}_v^f \lambda_{v-f}\, \mathrm{d}w^f. \tag{2}$$

Let xy be the oriented edge with vertices x, y, xyz the oriented triangle with vertices x, y, z and $xyzt$ the oriented volume with vertices x, y, z, t. In a code conceived in terms of proxy vector fields, with an underlying metric, instead of differential forms, the evaluation of circulations along edges (or fluxes across surfaces, etc.) of w^e (or w^f, etc.) is done according to the following well known result.

Proposition 2.2. *Let $v = \{k, l, m, n\}$ be a given tetrahedron. Then*

$$\begin{aligned}
&\langle w^n, x\rangle = \lambda_n(\mathbf{x}), \quad x \in v,\\
&\langle w^e, xy\rangle = |xy|\big(w^e(\mathbf{x}_{xy})\cdot \mathbf{t}_{xy}\big), \quad xy \subset v,\ e = \{m,n\},\\
&\langle w^f, xyz\rangle = |xyz|\big(w^f(\mathbf{x}_{xyz})\cdot \mathbf{n}_{xyz}\big), \quad xyz \subset v,\ f = \{m,n,k\},\\
&\langle w^v, xyzt\rangle = |xyzt|, \quad xyzt \subset v,
\end{aligned} \tag{3}$$

where $\mathbf{x}_{xy}$ (resp. $\mathbf{x}_{xyz}$) is the barycenter of xy (resp. xyz), $\mathbf{t}_{xy}$ (resp. $\mathbf{n}_{xyz}$) is the unit vector along xy (resp. normal to xyz), $|xy|$ is the length of xy, $|xyz|$ the area of xyz, and $|xyzt|$ the volume of $xyzt$.

Note that Proposition 2.2 relies on *metric tools*, such as dot product, segment lengths, etc., to compute *metric-free* quantities. The weight $\langle w^f, xyz\rangle$ does not depend, in fact, on the shape of f and xyz but on their relative position and orientation.

Thanks to formula (1), in the following Proposition we state an equivalent but *affine* way to compute the weights.

Proposition 2.3. *Let $v = \{k, l, m, n\}$ be a given tetrahedron. Then*

$$\langle w^n, x\rangle = \det\big(\lambda_n(\mathbf{x})\big), \quad x \in v,$$

$$\langle w^e, xy\rangle = \det\begin{pmatrix} \lambda_m(\mathbf{x}) & \lambda_n(\mathbf{x})\\ \lambda_m(\mathbf{y}) & \lambda_n(\mathbf{y})\end{pmatrix}, \quad xy \subset v,\ e = \{m,n\},$$

$$\langle w^f, xyz\rangle = \det\begin{pmatrix} \lambda_m(\mathbf{x}) & \lambda_n(\mathbf{x}) & \lambda_k(\mathbf{x})\\ \lambda_m(\mathbf{y}) & \lambda_n(\mathbf{y}) & \lambda_k(\mathbf{y})\\ \lambda_m(\mathbf{z}) & \lambda_n(\mathbf{z}) & \lambda_k(\mathbf{z})\end{pmatrix}, \quad xyz \subset v,\ f = \{m,n,k\},$$

$$\langle w^v, xyzt\rangle = \det\begin{pmatrix} \lambda_m(\mathbf{x}) & \lambda_n(\mathbf{x}) & \lambda_k(\mathbf{x}) & \lambda_l(\mathbf{x})\\ \lambda_m(\mathbf{y}) & \lambda_n(\mathbf{y}) & \lambda_k(\mathbf{y}) & \lambda_l(\mathbf{y})\\ \lambda_m(\mathbf{z}) & \lambda_n(\mathbf{z}) & \lambda_k(\mathbf{z}) & \lambda_l(\mathbf{z})\\ \lambda_m(\mathbf{t}) & \lambda_n(\mathbf{t}) & \lambda_k(\mathbf{t}) & \lambda_l(\mathbf{t})\end{pmatrix}, \quad xyzt \subset v.$$

Proof. The first statement is evident. For the second statement, by Definition 2.1, we can write $\langle w^e, xy\rangle = \sum_{n\in\mathcal{N}_\mathsf{m}} \mathbf{G}_e^n \lambda_{e-n}(\mathbf{x})\langle \mathrm{d}w^n, xy\rangle$. By duality between d and ∂, the equality $\langle \mathrm{d}w^n, xy\rangle = \langle w^n, \partial(xy)\rangle$ holds for any node $n \in \mathcal{N}_\mathsf{m}$ and for any segment $xy \subset v$. We remark that $\langle w^n, \partial(xy)\rangle = \langle w^n, y - x\rangle$. Since $0 = \langle w^e, xx\rangle$, we have

$$\langle w^e, xy\rangle = \sum_{n\in\mathcal{N}_\mathsf{m}} \mathbf{G}_e^n \lambda_{e-n}(\mathbf{x})\langle w^n, y\rangle = -\lambda_n(\mathbf{x})\lambda_m(\mathbf{y}) + \lambda_m(\mathbf{x})\lambda_n(\mathbf{y}) = \det\begin{pmatrix} \lambda_m(\mathbf{x}) & \lambda_n(\mathbf{x})\\ \lambda_m(\mathbf{y}) & \lambda_n(\mathbf{y})\end{pmatrix}.$$

For the third statement, by Definition 2.1, we can write $\langle w^f, xyz\rangle = \sum_{e\in\mathcal{E}_\mathsf{m}} \mathbf{R}_f^e \lambda_{f-e}(\mathbf{x})\langle \mathrm{d}w^e, xyz\rangle$. By duality between d and ∂, the equality $\langle \mathrm{d}w^e, xyz\rangle = \langle w^e, \partial(xyz)\rangle$ holds for any edge $e \in \mathcal{E}_\mathsf{m}$ and any surface $xyz \subset v$. We remark that $\langle w^e, \partial(xyz)\rangle = \langle w^e, xy + yz + zx\rangle$. Since $0 = \langle w^f, xzx\rangle = \sum_{e\in\mathcal{E}_\mathsf{m}} R_f^e \lambda_{f-e}(\mathbf{x})\langle w^e, zx\rangle$ and $0 = \langle w^f, xxy\rangle = \sum_{e\in\mathcal{E}_\mathsf{m}} R_f^e \lambda_{f-e}(\mathbf{x})\langle w^e, xy\rangle$, we have

$$\big\langle w^f, xyz\big\rangle = \sum_{e\in\mathcal{E}_\mathsf{m}} \mathbf{R}_f^e \lambda_{f-e}(\mathbf{x})\langle w^e, yz\rangle = \lambda_m(\mathbf{x})\big\langle w^{\{n,k\}}, yz\big\rangle + \lambda_n(\mathbf{x})\big\langle w^{\{k,m\}}, yz\big\rangle + \lambda_k(\mathbf{x})\big\langle w^{\{m,n\}}, yz\big\rangle$$

which proves the second statement. The same strategy can be applied to prove the last statement. □

In the following corollary, we provide the geometrical interpretation of the weights.

Corollary 2.4. *Let $v = \{k, l, m, n\}$ be a given tetrahedron with unit volume $|klmn|$. Then*

$$\begin{aligned}
&\langle w^n, x\rangle = |xklm|, \quad x \in v,\\
&\langle w^e, xy\rangle = |xykl|, \quad xy \subset v,\ e = \{m, n\},\\
&\langle w^f, xyz\rangle = |xyzl|, \quad xyz \subset v,\ f = \{m, n, k\},\\
&\langle w^v, xyzt\rangle = |xyzt|, \quad xyzt \subset v.
\end{aligned} \tag{4}$$

Proof. Let us denote by $\mathbf{k}$ the vector $(x_k, y_k, z_k)^t$ and similarly for $\mathbf{n}$, $\mathbf{m}$, $\mathbf{l}$, and $\mathbf{x}$. By definition of barycentric coordinates of a point $x \in v$ with respect to the vertices k, l, m, n of v, we can write

$$\mathbf{x} = \lambda_k(\mathbf{x})\mathbf{k} + \lambda_l(\mathbf{x})\mathbf{l} + \lambda_m(\mathbf{x})\mathbf{m} + \lambda_n(\mathbf{x})\mathbf{n}, \quad \text{with } 1 = \lambda_k(\mathbf{x}) + \lambda_l(\mathbf{x}) + \lambda_m(\mathbf{x}) + \lambda_n(\mathbf{x}). \tag{5}$$

By subtracting $\mathbf{k}$ from both sides, we get $\mathbf{x} - \mathbf{k} = \lambda_l(\mathbf{x})(\mathbf{l} - \mathbf{k}) + \lambda_m(\mathbf{x})(\mathbf{m} - \mathbf{k}) + \lambda_n(\mathbf{x})(\mathbf{n} - \mathbf{k})$. So, $\lambda_n(\mathbf{x}) = \det(\mathbf{l} - \mathbf{k}, \mathbf{m} - \mathbf{k}, \mathbf{x} - \mathbf{k}) / \det(\mathbf{l} - \mathbf{k}, \mathbf{m} - \mathbf{k}, \mathbf{n} - \mathbf{k}) = |xklm|/|nklm|$, being the mixed product $(\mathbf{l} - \mathbf{k}) \cdot [(\mathbf{m} - \mathbf{k}) \times (\mathbf{x} - \mathbf{k})] = 6|xklm|$ equal to $\det(\mathbf{l} - \mathbf{k}, \mathbf{m} - \mathbf{k}, \mathbf{x} - \mathbf{k})$.

Concerning the second statement, we write

$$\det\begin{pmatrix} \lambda_m(\mathbf{x}) & \lambda_n(\mathbf{x}) \\ \lambda_m(\mathbf{y}) & \lambda_n(\mathbf{y}) \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \lambda_k(\mathbf{x}) & \lambda_l(\mathbf{x}) & \lambda_m(\mathbf{x}) & \lambda_n(\mathbf{x}) \\ \lambda_k(\mathbf{y}) & \lambda_l(\mathbf{y}) & \lambda_m(\mathbf{y}) & \lambda_n(\mathbf{y}) \end{pmatrix} = \det\begin{pmatrix} \lambda_k(\mathbf{k}) & \lambda_l(\mathbf{k}) & \lambda_m(\mathbf{k}) & \lambda_n(\mathbf{k}) \\ \lambda_k(\mathbf{l}) & \lambda_l(\mathbf{l}) & \lambda_m(\mathbf{l}) & \lambda_n(\mathbf{l}) \\ \lambda_k(\mathbf{x}) & \lambda_l(\mathbf{x}) & \lambda_m(\mathbf{x}) & \lambda_n(\mathbf{x}) \\ \lambda_k(\mathbf{y}) & \lambda_l(\mathbf{y}) & \lambda_m(\mathbf{y}) & \lambda_n(\mathbf{y}) \end{pmatrix}.$$

Thanks to Proposition 2.3 and the change of basis (5) from barycentric coordinates to Cartesian ones, we have $\langle w^e, xy\rangle = |xykl|$. The same strategy is used to prove the last two statements. □

Corollary 2.4 is important for another aspect, namely, we can talk about 'volumes' $|xyzt|$ in a general way, without specifying the basis to compute it. A volume exists independently of the vector space basis, whereas the determinant of the matrix to compute it is always related to a basis.

3. Conclusions

We have presented some simple strategies to compute the weights of simplicial Whitney forms of polynomial degree one. These weights are *affine* invariants, they do not depend on metric quantities such as segment lengths or surface areas, but on relative positions and orientations. Proposition 2.3 shows that it is possible to compute these weights in an *affine* way which is equivalent to the classical *metric-dependent* one of Proposition 2.2. Affine strategies should be preferred in the framework of Whitney differential forms. Their utilization, for instance, in a multigrid context [7,3], makes the exchange of information between different discretization levels of easy access whatever the refinement procedure is used to generate these levels. Affine strategies make also possible to compute the weights for Whitney forms of higher order in a smart way. The technique presented in Proposition 2.2 can be used when dealing with differential forms of degree $r > 1$, provided that the integration rule is modified accordingly to be exact for polynomials of degree r. This problem is far from being trivial and is linked to another one, namely, the location in a p-simplex of the degrees of freedom associated with Whitney elements of order $r > 1$. Both problems will be addressed in future work.

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