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Dynamical Systems

On convergence of generalized continued fractions and Ramanujan's conjecture

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Abstract

We consider continued fractions

$$\cfrac{-a_1}{1-\cfrac{a_2}{1-\cfrac{a_3}{1-\cdots}}} \tag{CF}$$

with real coefficients a_i converging to a limit a. Ramanujan claimed that if $a \neq \frac{1}{4}$, then the fraction converges if and only if $a < \frac{1}{4}$. The statement of convergence was proved by Van Vleck in 1904 for complex a_i converging to $a \in \mathbb{C} \setminus [\frac{1}{4}, +\infty)$. Gill proved the divergence of (CF) under the assumption that $a_i \to a > \frac{1}{4}$ fast enough, more precisely, whenever $\sum_i |a_i - a| < \infty$. The Ramanujan's conjecture saying that (CF) always diverges whenever $a_i \to a > \frac{1}{4}$ remained, up to now, an open question. In the present Note we disprove it. We show that for any $a > \frac{1}{4}$ there exists a real sequence $a_i \to a$ such that (CF) converges. Moreover, we show that Gill's sufficient divergence condition is the optimal condition on the speed of convergence of the a_i's. ***To cite this article: A.A. Glutsyuk, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***
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Résumé

Sur convergence des fractions continues généralisées et une conjecture de Ramanujan. Nous considérons une fraction continue

$$\cfrac{-a_1}{1-\cfrac{a_2}{1-\cfrac{a_3}{1-\cdots}}} \tag{FC}$$

à coéfficients réels $a_i \to a$. Ramanujan a affirmé, que si $a \neq \frac{1}{4}$, alors la fraction converge, si et seulement si $a < \frac{1}{4}$. La convergence a été démontrée par Van Vleck en 1904 pour a_i complexes convergeant vers un $a \in \mathbb{C} \setminus [\frac{1}{4}, +\infty)$. Gill a démontré (en 1973), que la fraction diverge, si $a_i \to a > \frac{1}{4}$ assez vite, plus précisement, si $\sum_i |a_i - a| < \infty$.

La conjecture de Ramanujan disant que la fraction diverge toujours, quand $a > \frac{1}{4}$, restait ouverte jusqu'au présent. Nous montrons, qu'elle est fausse : pour tout $a > \frac{1}{4}$ il existe une suite réelle $a_i \to a$ telle que la fraction converge. Nous montrons aussi,

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que la condition précedante de Gill, qui est suffisante pour que la fraction diverge, est celle optimale sur la vitesse de convergence des a_i. ***Pour citer cet article : A.A. Glutsyuk, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***
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Version française abrégée

Théorème 0.1. *Pour tout $a > \frac{1}{4}$ il existe une suite $a_i \to a$, telle que la fraction* (1) *converge.*

Théorème 0.2. *Étant donnés un $q \in \mathbb{N}$, $q \geqslant 3$, et une suite $r_i \to 0$, $r_i > 0$, telle que $\sum_i r_i = \infty$. Alors il existe toujours un $a > \frac{1}{4}$ et une suite*

$$a_i \to a,\ a_i = a, \quad \text{si } i \not\equiv 1, 2 \pmod q, \quad a_{qi+1} - a = \mathrm{O}(r_i), \quad a_{qi+2} - a = \mathrm{O}(r_i), \quad \text{quand } i \to \infty,$$

telle que la fraction (1) *converge.*

1. Main results and the plan of the paper

We consider continued fractions

$$\cfrac{-a_1}{1 - \cfrac{a_2}{1 - \cfrac{a_3}{1 - \cdots}}} \tag{1}$$

with real coefficients a_i converging to a limit a. Ramanujan claimed (see [1], p. 38) that if $a \neq \frac{1}{4}$, then the fraction converges if and only if $a < \frac{1}{4}$. The statement of convergence was proved in [6] for complex a_i converging to $a \in \mathbb{C} \setminus [\frac{1}{4}, +\infty)$ (see also [4]). Gill [2] proved the divergence of (1) under the assumption that $a_i \to a > \frac{1}{4}$ fast enough, more precisely, whenever

$$\sum_i |a_i - a| < \infty. \tag{2}$$

The Ramanujan conjecture saying that (1) diverges always whenever $a_i \to a > \frac{1}{4}$ remained up to now an open question. In the present Note we disprove it. We show (Theorem 1.1) that for any $a > \frac{1}{4}$ there exists a real sequence $a_i \to a$ such that (1) converges.[1] Moreover, we show (Theorem 1.2) that Gill's sufficient divergence condition (2) is the optimal condition on the speed of convergence of the a_i's.

1.1. Main results

Theorem 1.1. *For any $a > \frac{1}{4}$ there exists a real sequence $a_i \to a$ such that the continued fraction* (1) *converges.*

The author did not find the Theorem in the literature. Its proof given below is elementary and it seems surprizing that it was not discovered before. In the case, when $a_i \to \frac{1}{4}$, fraction (1) may also converge or diverge, see paper [3] and its bibliography.

Theorem 1.2. *Let $q \in \mathbb{N}$, $q \geqslant 3$. Consider arbitrary sequence $r_i \to 0$, $r_i > 0$, such that $\sum_i r_i = \infty$. Then there exists an $a > \frac{1}{4}$ and a real sequence*

$$a_i \to a,\ a_i = a, \quad \text{if } i \not\equiv 1, 2 \pmod q, \quad a_{qi+1} - a = \mathrm{O}(r_i), \quad a_{qi+2} - a = \mathrm{O}(r_i), \quad \text{as } i \to \infty, \tag{3}$$

such that the continued fraction (1) *converges.*

[1] The author acknowledges that Alexey Tsygvintsev has constructed (by a completely different method) a beautiful explicit example [5] of a sequence $a_i \to 1$ corresponding to a convergent fraction (1) given by a simple recurrent formula. This example comes from the Analytic Function Theory.

1.2. The plan of the proofs and generalizations

In the proof of Theorem 1.1 the continued fraction (1) is expressed as a limit of compositions of Möbius transformations of the closed upper half-plane $\mathcal{H} = \{\operatorname{Im} z \geqslant 0\}$ of the type

$$T_b : \mathcal{H} \to \mathcal{H}, \quad T_b(z) = -\frac{b}{z+1}, \quad b \in \mathbb{R}.$$

Proposition 1.3. *The subsequent ratios $\frac{p_n}{q_n}$ of the continued fraction* (1) *are*

$$\frac{p_n}{q_n} = \tau_n = T_{a_1} \circ \cdots \circ T_{a_n}(0). \tag{4}$$

Proposition 1.3 is well-known and follows immediately from definition (by induction in n).
We recall the following

Definition 1.4. A Möbius transformation $M : \mathcal{H} \to \mathcal{H}$ of the upper half-plane is said to be *elliptic* (respectively, hyperbolic), if it has only one fixed point in $\operatorname{Int}(\mathcal{H})$ (respectively, two fixed points on the boundary of $\mathcal{H}$, one of them is an attractor, the other one is a repeller). (An elliptic transformation is Möbius conjugated to a rotation of the unit disc. By definition, its *rotation number* is $(2\pi)^{-1}$ times the corresponding rotation angle.) The *multiplier* of a hyperbolic transformation T, denoted by $\mu = \mu(T)$, is its derivative at the attractor (by definition, $0 < \mu < 1$).

Remark 1. The transformation T_a is elliptic if, and only if, $a > \frac{1}{4}$. Let $\rho(a)$ and $c(a)$ be the rotation number and the fixed point of T_a in $\mathcal{H}$, respectively. The functions $\rho : (\frac{1}{4}, +\infty) \to (0, \frac{1}{2})$ and $c : (\frac{1}{4}, +\infty) \to \mathrm{i}\mathbb{R}_+ - \frac{1}{2}$ are analytic diffeomorphisms. One has

$$\rho(1) = \frac{1}{3},$$

since T_1 permutes cyclically 0, 1 and ∞. If $\rho(a) = \frac{p}{q} \in \mathbb{Q}$, then $T_a^q = \mathrm{Id}$.

Example 1. Let $a_i \equiv a > \frac{1}{4}$. Then $\tau_n = T_a^n(0)$ does not have a limit: this sequence is either periodic or dense in $\mathbb{R}$.

Remark 2. By Gill's result [2], if a Möbius transformation sequence $\phi_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ converges to an elliptic transformation, then the composition values $\tau_n = \phi_1 \circ \cdots \circ \phi_n(0)$ converge, provided that the modules of multipliers at fixed points of ϕ_n do not converge to 1 too fast and the fixed points converge fast enough. This does not apply to our situation of real $a_i \to a > \frac{1}{4}$, for which $\phi_i = T_{a_i}$ are elliptic.

Theorem 1.5. *Theorem* 1.1 *holds for every a with $\rho(a) \in \mathbb{Q}$. Let $\{T_a\}_{a \in \mathbb{R}} : \mathcal{H} \to \mathcal{H}$ be arbitrary smooth family of elliptic transformations* (*depending on a parameter a*) *with a smooth fixed point family $c(a)$. Let a be a parameter value such that $\rho(a) = \frac{p}{q} \in \mathbb{Q} \setminus \{0, \frac{1}{2}\}$ and $c'(a) \neq 0$. Then there exists a sequence $a_i \to a$ such that the corresponding sequence τ_n from* (4) *converges. Moreover, for every divergent series $\sum_i r_i = \infty$, $r_i \to 0_+$, the sequence a_i can be chosen to satisfy* (3)*.*

Theorem 1.6. *Theorem* 1.1 *holds for each a with $\rho(a) \notin \mathbb{Q}$. Moreover, let T_a be arbitrary family of elliptic transformations as in Theorem* 1.5*. Let a be a parameter value such that $\rho(a) \notin \mathbb{Q}$ and $\rho \not\equiv const$ near a. Then there exists a sequence $a_i \to a$ such that τ_n converge.*

Theorems 1.5 and 1.6 imply Theorem 1.1. Theorem 1.5 implies Theorem 1.2. Theorem 1.5 is proved in the next section and Theorem 1.6 is proved in Section 3.

2. Limits with rational $\rho(a)$: proof of Theorem 1.5

Let $\rho(a) = \frac{p}{q} \in \mathbb{Q}$, thus, $T_a^q = 1$. We choose appropriate $\alpha_r, \beta_r \to a$ as specified below and set $a_i = a$ if $i \not\equiv 1, 2 \pmod q$; $a_{qr+1} = \alpha_r$, $a_{qr+2} = \beta_r$. Denote (5)

$T_{\alpha,\beta,q} = T_\alpha \circ T_\beta \circ T_a^{q-2}$, $\alpha, \beta \in \mathbb{R}$. We choose α_r, β_r so that

$T_{\alpha_r,\beta_r,q}$ are hyperbolic, let A_r, R_r be their attractors and repellers, respectively, (6)

$$A_i \to A, \quad R_i \to R, \quad A \neq R, \quad R \notin C_a = \left\{T_a^l(0),\ l = 0, \dots, q-1\right\}, \tag{7}$$

$$\prod_r \mu_r = 0, \quad \mu_r = \mu(T_{\alpha_r,\beta_r,q}). \tag{8}$$

The existence of α_r, β_r is proved at the end of this section.

We show in the next Proposition and the paragraph after that the corresponding sequence τ_n converges, whenever conditions (6)–(8) hold.

Proposition 2.1. *Let $H_1, H_2, \dots$ be an arbitrary sequence of hyperbolic transformations $\mathcal{H} \to \mathcal{H}$ of the upper half-plane, A_i, R_i be respectively their attractors and repellers, $A_i \to A$, $R_i \to R \neq A$. Let* (8) *hold with $\mu_r = \mu(H_r)$. Then the mapping sequence $\widehat{H}_n = H_1 \circ \cdots \circ H_n$ converges to a constant map uniformly on compact sets in $\partial\mathcal{H} \setminus R$.*

Proof. If $A_i = A$, $R_i = R$, then the proposition follows immediately. If we fix a compact set $K \Subset \partial\mathcal{H} \setminus \{A \cup R\}$, then for any i large enough the transformation H_i moves the points of K towards A (along $\partial\mathcal{H}$) by asymptotically the same distance, as the hyperbolic transformation with the same multiplier μ_i but with $A_i = A$, $R_i = R$. This together with the previous statement and the monotonicity of the restrictions $H_i|_{\partial\mathcal{H}}$ implies the claim. □

2.1. Proof of convergence of τ_n

If (6)–(8) hold, then the transformations $H_r = T_{\alpha_r,\beta_r,q}$ satisfy the conditions of Proposition 2.1. Hence, their compositions $\widehat{H}_r = T_{a_1} \circ T_{a_2} \circ \cdots \circ T_{a_{qr}}$ converge uniformly to a constant limit (denote it x) on compact sets in $\partial\mathcal{H} \setminus R$. By definition, $\tau_{qr} = \widehat{H}_r(0)$, $0 \neq R$ by (7). This implies that $\tau_{qr} \to x$, as $r \to \infty$. To show that the whole sequence τ_n converges to x, we use condition (7), which says that the finite T_a-orbit C_a of 0 does not meet R. Let $\delta > 0$ be such that the closed δ-neighborhood U of the latter orbit be disjoint from R. Then $\widehat{H}_r \to x$ uniformly on U. If r is large enough, then

$$T_{\alpha_r}(0), T_{\alpha_r} \circ T_{\beta_r}(0), T_{\alpha_r} \circ T_{\beta_r} \circ T_a^{l-2}(0) \in U \ (0 < l \leqslant q).$$

By definition, $\tau_{qr+1} = \widehat{H}_r \circ T_{\alpha_r}(0)$, $\quad \tau_{qr+l} = \widehat{H}_r \circ T_{\alpha_r} \circ T_{\beta_r} \circ T_a^{l-2}(0)$.

The two last statements imply together that the q sequences $\tau_{qr}, \tau_{qr+1}, \tau_{qr+2}, \dots, \tau_{qr+q-1}$ converge to x, hence, the whole sequence τ_n converges. Below we show that for arbitrary given $r_i > 0$, $\sum_i r_i = \infty$, one can achieve that in addition to (6)–(8), condition (3) holds. This will prove Theorems 1.5 and 1.2.

2.2. The construction of sequences α_i, β_i satisfying (6)–(8)

Lemma 2.2 (Main Technical Lemma). *Let a family T_a and a parameter value a be as in Theorem* 1.5. *Then for any point $R \in \partial\mathcal{H}$* (*may be except two points*) *there exist two linear families of parameter values*

$$\alpha(t) = a + c_1 t, \quad \beta(t) = a + c_2 t, \quad c_1, c_2 \in \mathbb{R}, \tag{9}$$

such that for any $t > 0$ small enough the transformation $T_{\alpha(t),\beta(t),q} = T_{\alpha(t)} \circ T_{\beta(t)} \circ T_a^{q-2}$ be hyperbolic and its repeller $R(t)$ (*respectively, attractor $A(t)$*) *tends to R* (*respectively, to a point $A \neq R$*), *as $t \to 0_+$. Moreover, one can achieve that the families $A(t)$, $R(t)$ be smooth at 0, and the derivative in t at $t = 0$ of the previous family $T_{\alpha(t),\beta(t),q}$ be nonzero.*

The lemma is proved below.

Let $\rho(a) = \frac{p}{q}$, C_a be the (finite) T_a-orbit of 0. Let us choose any $R \notin C_a$ that satisfies the statements of Lemma 2.2. Let $\alpha(t)$, $\beta(t)$ be the corresponding families (9). Take a sequence $t_k \to 0_+$ and put $\alpha_k = \alpha(t_k)$, $\beta_k = \beta(t_k)$. The conditions (6) and (7) follow immediately from construction. Condition (8) holds, if and only if $\sum t_i = \infty$ (these are the t_i we choose). This follows from the fact that the function $\mu(t)=\mu(T_{\alpha(t),\beta(t),q})$ has nonzero derivative at 0: $\mu(t)=1+st+\mathrm{O}(t^2)$, $s\neq 0$, hence, $\ln\mu_k=st_k(1+\mathrm{o}(1))$. Indeed, otherwise, if $\mu'(0)=0$, the transformation family $T_{\alpha(t),\beta(t),q}$ would have zero derivative in t at $t=0$ – a contradiction to the hypotheses of the lemma. This finishes the construction. Statement (3) follows immediately, if we put $t_i=r_i$. Theorem 1.5 is proved.

Proof of Lemma 2.2. Consider $T_{\alpha,\beta,q}=T_\alpha\circ T_\beta\circ T_a^{q-2}$ as a family of mappings depending on two parameters α and β (the a is fixed). It is identity, if $\alpha,\beta=a$. Consider its derivatives in α (β) at $(\alpha,\beta)=(a,a)$ as vector fields on $\partial\mathcal{H}$ denoted v_1 (respectively, v_2). It follows from definition that $v_2=(T_a)_*v_1$. We claim that the fields v_1 and v_2 are not constant-proportional. Indeed, otherwise the group generated by T_a and the 1-parametric subgroup in $\mathrm{Aut}(\mathcal{H})$ generated by v_1 would be solvable. Since T_a is elliptic, this implies that T_a either belongs to the same 1-parameter subgroup, or is an involution. The first case is impossible: otherwise $c'(a)=0$ – a contradiction to the conditions of the Lemma. The second case is impossible by the hypothesis $\rho(a)\neq 0,\frac{1}{2}$.

Thus, the vector fields v_1 and v_2 are not proportional. Hence, for any point $R\in\partial\mathcal{H}$ one can find a linear combination $v=c_1v_1+c_1v_2\not\equiv 0$ that vanishes at R. If the 1-jet of v at R does not vanish (then one can achieve that $v'(R)>0$ by changing sign), this implies that v has another zero $A\in\partial\mathcal{H}\setminus R$. Then the corresponding families (9) are those we are looking for. If the latter 1-jet vanishes, this implies that the commutator $[v_1,v_2]$ (which also belongs to the Lie algebra of the group $\mathrm{Aut}(\mathcal{H})$) vanishes at R. The latter commutator does not vanish identically (since v_i are not proportional) and cannot have more than two zeros. This together with the previous discussion proves the lemma. □

3. Case of irrational limit rotation

Proof of Theorem 1.6. Let $\rho(a)\notin\mathbb{Q}$, $\tilde a_n\to a$, $\rho(\tilde a_n)=\frac{p_n}{q_n}\in\mathbb{Q}$. We choose appropriate $\alpha_n,\beta_n\to a$, a natural number sequence $N_1,N_2,\dots$, and define a_n as follows:

1) For $n\leqslant N_1q_1$ set $a_n=\tilde a_1$, if $n\not\equiv 1,2 \pmod{q_1}$; $a_{q_1r+1}=\alpha_1$, $a_{q_1r+2}=\beta_1$, $r<N_1$.

2) Let $N_1q_1<n\leqslant N_1q_1+N_2q_2$. Put $n_1=n-N_1q_1$, $a_n=\tilde a_2$, if $n_1\not\equiv 1,2 \pmod{q_2}$; $a_n=\alpha_2$, if $n_1\equiv 1\pmod{q_2}$; $a_n=\beta_2$, if $n_1\equiv 2\pmod{q_2}$, etc.

We show that (1) converges if we take α_i, β_i and N_i as specified below.

Choice of α_n and β_n. Denote

$\psi_0=\psi_{n,0}=\mathrm{Id}$, $\psi_n=T_{\alpha_n,\beta_n,q_n}=T_{\alpha_n}\circ T_{\beta_n}\circ T_{\tilde a_n}^{q_n-2}$, $\psi_{n,1}=T_{\alpha_n}$, $\psi_{n,2}=\psi_{n,1}\circ T_{\beta_n}$, $\psi_{n,l}=\psi_{n,2}\circ T_{\tilde a_n}^{l-2}$, for $2\leqslant l\leqslant q_n-1$. We choose α_n and β_n so that the transformations

$$\psi_n \text{ be hyperbolic, denote } A_n,\ R_n \text{ their attractors and repellers, respectively,} \tag{10}$$

$$R_n\notin M_n=\bigl\{A_{n+1},\psi_{n+1,l}(A_{n+1}),0,\psi_{n+1}^r\circ\psi_{n+1,l}(0)\mid 0\leqslant l\leqslant q_{n+1}-1,\ r\in\mathbb{N}\cup 0\bigr\}. \tag{11}$$

Remark 3. The set M_n in (11) is infinite and accumulates exactly to the finite $T_{\tilde a_n}$-orbit of A_{n+1}, which follows from definition. This implies that if (11) holds, then M_n does not accumulate to R_n. Thus, in this case choosing appropriate power N_n, one can achieve that the image $\psi_n^{N_n}(M_n)$ be arbitrarily close to A_n.

Choice of N_i. The parameters α_i, β_i satisfy (10), (11). Put $\theta_k=\psi_1^{N_1}\circ\cdots\circ\psi_k^{N_k}$. We construct N_i (by induction in i) in such a way that

$$\mathrm{diam}\bigl(\theta_k(M_k)\bigr)<\frac{1}{2^k}. \tag{12}$$

It is possible by the last statement of the previous remark. Let us show that then the sequence τ_n is Cauchy (hence, converges). Denote $n_k=\sum_{i=1}^k q_iN_i$. It suffices to show that

$$\text{for any } k \text{ and any } m\geqslant n_k \text{ one has } \mathrm{dist}(\tau_{n_k},\tau_m)<\frac{1}{2^{k-2}}. \tag{13}$$

Case 1: $m=n_i>n_k$, say, $m=n_{k+1}$. Then $\tau_{n_k}=\theta_k(0)$, $\tau_m=\theta_{k+1}(0)=\theta_k\circ\psi_{k+1}^{N_{k+1}}(0)$. By definition, $0,\psi_{k+1}^{N_{k+1}}(0)\in M_k$. By (12), $\mathrm{dist}(\theta_k(0),\theta_{k+1}(0))=\mathrm{dist}(\theta_k(0),\theta_k(\psi_{k+1}^{N_{k+1}}(0)))<\frac{1}{2^k}$. Therefore,

$$\operatorname{dist}\bigl(\theta_k(0), \theta_s(0)\bigr) < \frac{1}{2^{k-1}} \quad \text{for any } s > k. \tag{14}$$

This proves (13) for any $m = n_i > n_k$.

General case: $m > n_k$ is arbitrary. Take $s \in \mathbb{N}$ such that $n_s \leqslant m < n_{s+1}$. Then $m = n_s + rq_{s+1} + l$, $0 \leqslant l < q_{s+1}$, $\tau_m = \theta_s \circ \psi_{s+1}^{r-1} \circ \psi_{s+1,l}(0)$. Analogously to the previous discussion, by (12), $\operatorname{dist}(\theta_s(0), \tau_m) < \frac{1}{2^s}$. This together with (14) implies (13). Theorem 1.6 is proved. □

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