Available online at www.sciencedirect.com
SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 341 (2005) 375–380

COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Numerical Analysis/Partial Differential Equations

On the numerical solution of a two-dimensional Pucci's equation with Dirichlet boundary conditions: a least-squares approach

Edward J. Dean [a], Roland Glowinski [b,a]

[a] *Department of Mathematics, University of Houston, Houston, TX 77024-3008, USA*
[b] *Laboratoire Jacques-Louis Lions, université Pierre et Marie Curie, 4, place Jussieu, 75005 Paris, France*

Received 29 July 2005; accepted 8 August 2005

Presented by Philippe G. Ciarlet

Abstract

In this Note we discuss the numerical solution of a two-dimensional, fully nonlinear elliptic equation of the Pucci's type, completed by Dirichlet boundary conditions. The solution method relies on a least-squares formulation taking place in a subset of $H^2(\Omega) \times \mathbf{Q}$, where $\mathbf{Q}$ is the space of the 2×2 symmetric tensor-valued functions with components in $L^2(\Omega)$. After an appropriate space discretization the resulting finite dimensional problem is solved by an iterative method operating alternatively in the spaces V_h and $\mathbf{Q}_h$ approximating $H^2(\Omega)$ and $\mathbf{Q}$, respectively. The results of numerical experiments are presented; they validate the methodology discussed in this Note. ***To cite this article: E.J. Dean, R. Glowinski, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Sur la solution numérique de l'équation bi-dimensionelle de Pucci avec conditions limites de Dirichlet : une formulation par moindres carrés. Dans cette Note, on étudie la résolution numérique d'une équation elliptique bi-dimensionelle, pleinement non linéaire et de type Pucci. La méthode de résolution repose sur une formulation par moindres carrés dans un sous-ensemble de $H^2(\Omega) \times \mathbf{Q}$ où $\mathbf{Q}$ est l'espace des fonctions à valeurs tensorielles symetriques 2×2, dont les composantes sont dans $L^2(\Omega)$. Après approximation par éléments finis, on résoud le problème en dimension finie qui en résulte par une méthode itérative qui opère alternativement dans les espaces V_h et $\mathbf{Q}_h$, approximations respectives de $H^2(\Omega)$ et $\mathbf{Q}$. Les résultats d'expériences numériques sont presentés ; ils valident la méthodologie numérique décrite dans cette Note. ***Pour citer cet article : E.J. Dean, R. Glowinski, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Problem formulations

Let Ω be a bounded domain of $\mathbf{R}^2$; we denote by Γ the boundary of Ω and by $x = \{x_1, x_2\}$ the generic point of $\mathbf{R}^2$. Following, e.g., Caffarelli and Cabré ([3]; see also the references therein and [2]) we consider the following *nonlinear Dirichlet problem* for the *Pucci's equation*: *Find* ψ *such that*

$$\alpha\lambda^+ + \lambda^- = 0 \quad \text{in } \Omega, \quad \psi = g \quad \text{on } \Gamma, \tag{PE-D}$$

E-mail addresses: dean@math.uh.edu (E.J. Dean), roland@math.uh.edu (R. Glowinski).

1631-073X/$ – see front matter © 2005 Published by Elsevier SAS on behalf of Académie des sciences.
doi:10.1016/j.crma.2005.08.002

where, in (PE-D): (i) λ^+ (resp., λ^-) denotes the *largest* (resp., the *smallest*) *eigenvalue* of the *Hessian matrix* $D^2\psi = (\partial^2\psi/\partial x_i\partial x_j)_{1\leqslant i,j\leqslant 2}$, (ii) $\alpha \in (1, +\infty)$ (if $\alpha = 1$, (PE-D) reduces to the Poisson–Dirichlet problem $\Delta\psi = 0$ in Ω, $\psi = g$ on Γ). We have thus $\lambda^+ = 1/2(\Delta\psi + (|\Delta\psi|^2 - 4\det D^2\psi)^{1/2})$ and $\lambda^- = 1/2(\Delta\psi - (|\Delta\psi|^2 - 4\det D^2\psi)^{1/2})$, which, combined with (PE-D), implies in turn that

$$(\alpha+1)\Delta\psi + (\alpha-1)\big(|\Delta\psi|^2 - 4\det D^2\psi\big)^{1/2} = 0 \quad \text{in } \Omega. \tag{1}$$

It follows then from (1) that problem (PE-D) is equivalent to

$$\begin{cases} \alpha|\Delta\psi|^2 + (\alpha-1)^2\det D^2\psi = 0 \quad \text{in } \Omega, \quad \psi = g \quad \text{on } \Gamma, \\ \Delta\psi \leqslant 0 \quad \text{in } \Omega. \end{cases} \tag{2}$$

Relations (2) show that the Pucci's problem discussed here combines (nonlinearly) Poisson and Monge–Ampère equations. The numerical solution of (PE-D), via (2), will be discussed in the following sections. Actually, assuming that $g \in H^{3/2}(\Gamma)$, we will look for solutions of (PE-D), (2) belonging to $H^2(\Omega)$.

2. Some exact solutions

In order to validate numerical solution methods it is always useful to have access to (nontrivial) exact solutions. Let $x_0 \in \mathbf{R}^2$; we shall denote $|x - x_0|$ by ρ. Suppose that u is a function of ρ only verifying the partial differential equation in (2). We have then (away from $x = x_0$ and with obvious notation)

$$\alpha\left|\rho^{-1}(\rho u')'\right|^2 + (\alpha-1)^2\rho^{-1}u'u'' = 0. \tag{3}$$

It follows from (3) that u defined by $u(x) = C\rho^m + p(x)$, where C is a constant, $m = 1 - \frac{1}{\alpha}$ or $1 - \alpha$ and p is a polynomial of degree $\leqslant 1$, is solution of the partial differential equation in (2). However, since $\Delta(\rho^m) = m^2\rho^{m-2}$ away from $x = x_0$, in order to verify the inequality in (2) we have to take $C < 0$. In other words, ψ defined by

$$\psi(x) = -C\rho^m + p(x), \tag{4}$$

with C a positive constant and m and p as above, verifies the partial differential equation and inequality in (2). If $x_0 \notin \overline{\Omega}$ then ψ defined by (4) belongs to $C^\infty(\overline{\Omega})$; on the other hand, if $x_0 \in \overline{\Omega}$ the above function ψ does not have the $H^2(\Omega)$-regularity.

3. A least-squares formulation of problem (2)

Problem (2) is clearly equivalent to

$$\begin{cases} \mathbf{p} = D^2\psi, \\ \alpha(p_{11} + p_{22})^2 + (\alpha-1)^2(p_{11}p_{22} - p_{12}^2) = 0, \quad p_{11} + p_{22} \leqslant 0, \\ \psi = g \quad \text{on } \Gamma, \end{cases} \tag{5}$$

with $\mathbf{p} = \mathbf{p}^t = (p_{ij})_{1\leqslant i,j\leqslant 2}$ and $p_{ij} = \partial^2\psi/\partial x_i\partial x_j$. Suppose that problem (2) has a solution in $H^2(\Omega)$. Following a strategy which has been successful with the Monge–Ampère equation (see [4]) we are going to investigate a least-squares method, operating in $H^2(\Omega)$ and related functional spaces, for the solution of problem (5). Let us introduce the following spaces and set:

$$V_g = \{\varphi \mid \varphi \in H^2(\Omega),\ \varphi = g \text{ on } \Gamma\}, \tag{6}$$

$$\mathbf{Q} = \{\mathbf{q} \mid \mathbf{q} = (q_{ij})_{1\leqslant i,j\leqslant 2},\ q_{ij} \in L^2(\Omega),\ \mathbf{q} = \mathbf{q}^t\}, \tag{7}$$

$$\mathbf{Q}_P = \{\mathbf{q} \mid \mathbf{q} \in \mathbf{Q},\ \alpha(q_{11} + q_{22})^2 + (\alpha-1)^2(q_{11}q_{22} - q_{12}^2) = 0,\ q_{11} + q_{22} \leqslant 0 \text{ a.e. in } \Omega\}. \tag{8}$$

The space $\mathbf{Q}$ is an Hilbert space for the following scalar product and norm:

$$(\mathbf{q}, \mathbf{q}')_{\mathbf{Q}} = \int_\Omega \mathbf{q} : \mathbf{q}'\,\mathrm{d}x \quad \text{and} \quad \|\mathbf{q}\|_{\mathbf{Q}} = \sqrt{(\mathbf{q}, \mathbf{q})_{\mathbf{Q}}} \quad \left(= \sqrt{\int_\Omega |\mathbf{q}|^2\,\mathrm{d}x}\right); \tag{9}$$

in (9), $\mathbf{S}:\mathbf{T} = s_{11}t_{11} + s_{22}t_{22} + s_{12}t_{12}$, $\mathbf{S} = (s_{ij})_{1\leqslant i,j\leqslant 2}$ and $\mathbf{T} = (t_{ij})_{1\leqslant i,j\leqslant 2}$, with $\mathbf{S} = \mathbf{S}^t$ and $\mathbf{T} = \mathbf{T}^t$, and $|\mathbf{S}| = \sqrt{\mathbf{S}:\mathbf{S}}$, $\forall \mathbf{S}$, $\mathbf{S} = \mathbf{S}^t$. A quite natural *least-squares formulation* of problem (5) reads as follows:

$$\begin{cases} \{\psi, \mathbf{p}\} \in V_g \times \mathbf{Q}_P, \\ j(\psi, \mathbf{p}) \leqslant j(\varphi, \mathbf{q}), \quad \forall \{\varphi, \mathbf{q}\} \in V_g \times \mathbf{Q}_P, \end{cases} \tag{LS.PE-D}$$

with

$$j(\varphi, \mathbf{q}) = \frac{1}{2}\int_\Omega |D^2\varphi - \mathbf{q}|^2\,\mathrm{d}x. \tag{10}$$

The *iterative solution* of problem (LS.PE-D) will be discussed in the following section.

4. Iterative solution of the least-squares problem

Let us denote by $I_{\mathbf{Q}_P}$ the *indicator functional* of the set $\mathbf{Q}_P$, namely, the mapping from $\mathbf{Q}$ into $\mathbf{R} \cup \{+\infty\}$ defined by $I_{\mathbf{Q}_P}(\mathbf{q}) = 0$ if $\mathbf{q} \in \mathbf{Q}_P$, $I_{\mathbf{Q}_P}(\mathbf{q}) = +\infty$ if $\mathbf{q} \in \mathbf{Q} \setminus \mathbf{Q}_P$. Problem (LS.PE-D) is clearly *equivalent* to

$$\min_{\{\varphi,\mathbf{q}\}\in V_g\times\mathbf{Q}} \big[j(\varphi, \mathbf{q}) + I_{\mathbf{Q}_P}(\mathbf{q}) \big]. \tag{11}$$

At $\{\psi, \mathbf{p}\}$ a *necessary optimality condition* for problem (11) reads as follows:

$$\begin{cases} \{\psi, \mathbf{p}\} \in V_g \times \mathbf{Q}; \quad \forall \{\varphi, \mathbf{q}\} \in V_0 \times \mathbf{Q}, \ \text{we have} \\ \displaystyle\int_\Omega (D^2\psi - \mathbf{p}) : (D^2\varphi - \mathbf{q})\,\mathrm{d}x + \langle \partial I_{\mathbf{Q}_P}(\mathbf{p}), \mathbf{q} \rangle = 0, \end{cases} \tag{12}$$

with $\partial I_{\mathbf{Q}_P}(\mathbf{p})$ a *generalized differential* of functional $I_{\mathbf{Q}_P}(\cdot)$ at $\mathbf{p}$. To (12), we associate the following *initial value problem*:

$$\begin{cases} \text{Find } \big\{\psi(t), \mathbf{p}(t)\big\} \in V_g \times \mathbf{Q}, \ \forall t \in (0, +\infty), \ \text{such that} \\ \displaystyle\int_\Omega \Delta(\partial\psi/\partial t) : \Delta\varphi\,\mathrm{d}x + \int_\Omega D^2\psi : D^2\varphi\,\mathrm{d}x = \int_\Omega \mathbf{p} : D^2\varphi\,\mathrm{d}x, \quad \forall \varphi \in V_0, \\ \displaystyle\int_\Omega \frac{\partial \mathbf{p}}{\partial t} : \mathbf{q}\,\mathrm{d}x + \int_\Omega \mathbf{p} : \mathbf{q}\,\mathrm{d}x + \langle \partial I_{\mathbf{Q}_P}(\mathbf{p}), \mathbf{q} \rangle = \int_\Omega D^2\psi : \mathbf{q}\,\mathrm{d}x, \quad \forall \mathbf{q} \in \mathbf{Q}, \\ \big\{\psi(0), \mathbf{p}(0)\big\} = \{\psi_0, \mathbf{p}_0\}. \end{cases} \tag{13}$$

In order to solve problem (13), we advocate *operator-splitting*; applying to the solution of (13) the *Marchuk–Yanenko scheme*, we obtain (with τ (>0) a time-discretization step):

$$\{\psi^0, \mathbf{p}^0\} = \{\psi_0, \mathbf{p}_0\}; \tag{14}$$

then for $n \geqslant 0$, $\{\psi^n, \mathbf{p}^n\}$ being known, compute $\{\psi^{n+1}, \mathbf{p}^{n+1}\}$ via the solution of

$$(\mathbf{p}^{n+1} - \mathbf{p}^n)/\tau + \mathbf{p}^{n+1} + \partial I_{\mathbf{Q}_P}(\mathbf{p}^{n+1}) = D^2\psi^n, \quad \text{and,} \tag{15}$$

$$\int_\Omega \Delta\big[(\psi^{n+1} - \psi^n)/\tau\big] : \Delta\varphi\,\mathrm{d}x + \int_\Omega D^2\psi^{n+1} : D^2\varphi\,\mathrm{d}x = \int_\Omega \mathbf{p}^{n+1} : D^2\varphi\,\mathrm{d}x, \quad \forall \varphi \in V_0. \tag{16}$$

Since *linear variational problems* such as (16) have been encountered already, when addressing for example the solution of the elliptic Monge–Ampère equation by augmented Lagrangians and least-squares methods (see [4,5] for details), we shall focus (in Section 5) on the solution of the (highly) *nonlinear problems* (15).

Remark 1. An alternative to scheme (14)–(16) is provided by

$$\{\psi^0, \mathbf{p}^0\} = \{\psi_0, \mathbf{p}_0\}; \tag{17}$$

then for $n \geqslant 0$, from $\{\psi^n, \mathbf{p}^n\}$ compute $\{\psi^{n+1}, \mathbf{p}^{n+1}\}$ via the solution of

$$(\mathbf{p}^{n+1/2} - \mathbf{p}^n)/\tau + \mathbf{p}^{n+1/2} + \partial I_{\mathbf{Q}_P}(\mathbf{p}^{n+1/2}) = \mathbf{0}, \tag{18}$$

$$\psi^{n+1} \in V_g; \quad \int_\Omega \Delta\big[(\psi^{n+1} - \psi^n)/\tau\big] : \Delta\varphi \,\mathrm{d}x + \int_\Omega D^2\psi^{n+1} : D^2\varphi \,\mathrm{d}x = \int_\Omega \mathbf{p}^{n+1/2} : D^2\varphi \,\mathrm{d}x, \quad \forall \varphi \in V_0, \tag{19}$$

$$(\mathbf{p}^{n+1} - \mathbf{p}^{n+1/2})/\tau = D^2\psi^{n+1}. \tag{20}$$

Other splitting schemes are possible.

5. Solution of the nonlinear problems (15)

Relation (15) is nothing but a *necessary optimality condition* for the following minimization problem:

$$\min_{\mathbf{q}\in\mathbf{Q}_P} \left[\frac{1}{2}(1+\tau)\int_\Omega |\mathbf{q}|^2 \,\mathrm{d}x - \int_\Omega (\mathbf{p}^n + \tau D^2\psi^n) : \mathbf{q} \,\mathrm{d}x\right]. \tag{21}$$

Problem (21) can be solved *point-wise* (in practice at the vertices of a finite element or finite difference mesh). Indeed, we have to minimize, a.e. on Ω, a three-variable polynomial of the following type $\frac{1}{2}(z_1^2 + z_2^2 + z_3^2) - (b_1z_1 + b_2z_2 + b_3z_3)$, over the set $\{\mathbf{z} \mid \mathbf{z} = \{z_i\}_{i=1}^3,\ \alpha|z_1 + z_2|^2 + (\alpha-1)^2(z_1z_2 - z_3^2) = 0,\ z_1 + z_2 \leqslant 0\}$. The above *three-dimensional problem* can be reduced to a simple *one-dimensional* one; to achieve this dimension reduction we shall proceed as follows:

(i) Denote $\alpha/(\alpha-1)^2$ by γ and observe that the above minimization problem is equivalent to the minimization of $\frac{1}{2}[z_1^2 + z_2^2 + \gamma(z_1+z_2)^2 + z_1z_2] - b_1z_1 - b_2z_2 - |b_3|(\gamma(z_1+z_2)^2 + z_1z_2)^{1/2}$ over the subset of $\mathbf{R^2}$ defined by $\{\{z_1, z_2\} \mid z_1 + z_2 \leqslant 0,\ \gamma(z_1+z_2)^2 + z_1z_2 \geqslant 0\}$ (completed by $z_3 = \mathrm{sign}(b_3)(\gamma(z_1+z_2)^2 + z_1z_2)^{1/2}$).

(ii) Take $z_1 = \rho\cos\theta$, $z_2 = \rho\sin\theta$, with $\rho \geqslant 0$ and $\theta \in [0, 2\pi)$. There is equivalence between the minimization problem in (i) and the maximization problem below

$$\max_{\theta\in K_\theta} F(\theta), \tag{22}$$

with $F(\theta) = [b_1\cos\theta + b_2\sin\theta + |b_3|[\gamma + (\frac{1}{2}+\gamma)\sin 2\theta]^{1/2}]\,/[1+\gamma+(\frac{1}{2}+\gamma)\sin 2\theta]^{1/2}$, $K_\theta = [\pi - \frac{1}{2}\varphi_c, \frac{3\pi}{2} + \frac{1}{2}\varphi_c]$ and $\varphi_c = \sin^{-1}[2\gamma/(2\gamma+1)]$. Let denote by θ_M the solution of problem (22); if $F(\theta_M) \leqslant 0$, the solution of the minimization problem (i) is $\{0, 0, 0\}$. If $F(\theta_M) > 0$, the solution of the above problem is $\mathbf{z} = \{z_{1M}, z_{2M}, z_{3M}\}$ with $z_{1M} = \rho_M\cos\theta_M$, $z_{2M} = \rho_M\sin\theta_M$, $z_{3M} = \mathrm{sign}(b_3)[\gamma(z_{1M} + z_{2M})^2 + z_{1M}z_{2M}]^{1/2}$, ρ_M being given by $\rho_M = [b_1\cos\theta_M + b_2\sin\theta_M + |b_3|[\gamma + (\frac{1}{2}+\gamma)\sin 2\theta_M]^{1/2}]\,/[1+\gamma+(\frac{1}{2}+\gamma)\sin 2\theta_M]$. To solve the maximization problem (22) we used the derivative-free methods discussed in [1].

6. On the initialization of algorithm (14)–(16)

Concerning the *initialization* of algorithm (14)–(16) (and (17)–(20)) an obvious choice is provided by $-\Delta\psi^0 = 0$ in Ω, $\psi^0 = g$ on Γ, followed by $\mathbf{p}^0 = D^2\psi^0$. A more sophisticated one (inspired by relation (1)) is the following: (i) Solve the following Poisson problem: $-\Delta\psi^{-1} = 0$ in Ω, $\psi^{-1} = g$ on Γ, and define $\mathbf{p}^{-1}$ by $\mathbf{p}^{-1} = D^2\psi^{-1}$. (ii) Solve $-\Delta\psi^0 = 2[(\alpha-1)/(\alpha+1)]\sqrt{|\det\mathbf{p}^{-1}|}$ in Ω, $\psi^0 = g$ on Γ and define $\mathbf{p}^0$ by $\mathbf{p}^0 = D^2\psi^0$.

7. Numerical experiments

Problem (PE-D), (2) being clearly of the Monge–Ampère type (albeit more complicated) we have used to approximate it the mixed finite element method discussed in [4–6]. Moreover, the results presented below have been obtained by a discrete variant of algorithm (17)–(20), since, on the basis of numerical experiments, this algorithm appears more robust and faster than (14)–(16). For the two families of test problems discussed below we have taken $\Omega = (0,1)\times(0,1)$ and defined the mixed finite element approximation, mentioned just above, from uniform triangulations, like those used in [4] and [5]. The *first family of test problems* is motivated by Section 2; for $\alpha \in [2,3]$ we consider those particular cases of problem (PE-D), (2) where the function g is the trace on Γ of the function $x \to -\rho^{1-\alpha}$ with $\rho = [(x_1+1)^2 + (x_2+1)^2]^{1/2}$. The above problem has $\psi = -\rho^{1-\alpha}$ as exact solution; we clearly have $\psi \in C^\infty(\overline{\Omega})$. Applying to problem (PE-D), (2) the solution method briefly discussed in the preceding sections we obtain the results shown in Table 1.

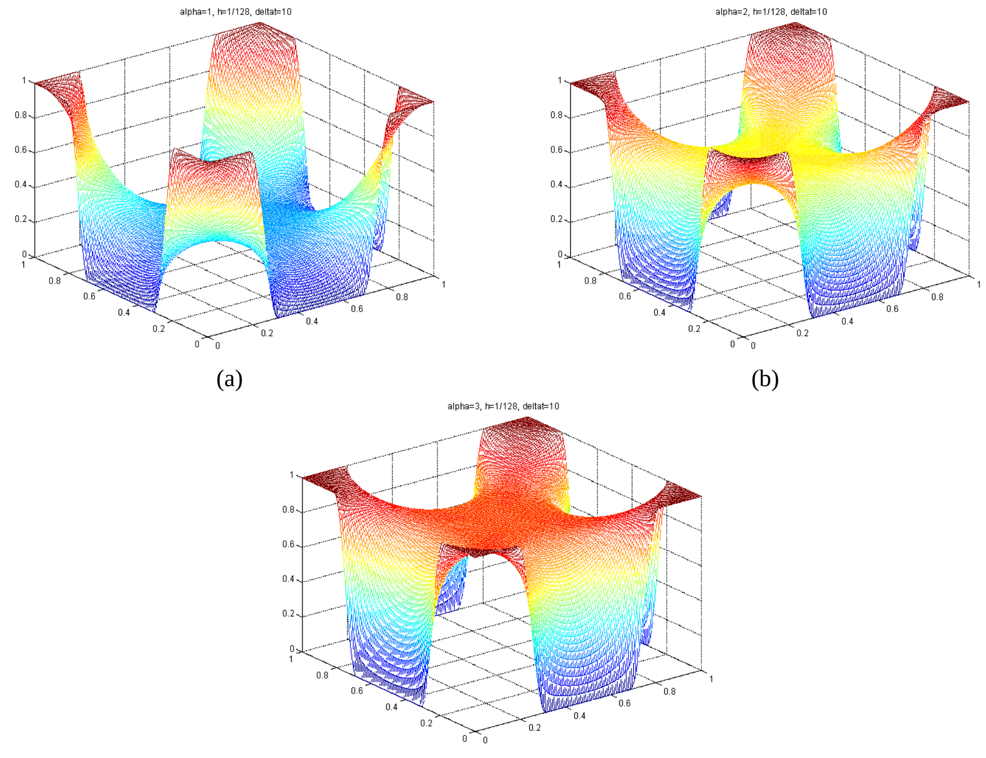
In Table 1, n_{it} denotes the number of iterations necessary to achieve convergence, the corresponding stopping criterion being $\|D_h^2\psi_h^n - \mathbf{p}_h^n\|_{0,\Omega} \leqslant \epsilon$ (with $\|\cdot\|_{0,\Omega}$ denoting the $L^2(\Omega)$-norm, the other notation being obvious);

Table 1
First test problem: convergence of the approximate solutions

α	h	τ	n_{it}	$\|\psi_h^c - \psi\|_{0,\Omega}$	$\|D_h^2\psi_h^c - \mathbf{p}_h^c\|_{0,\Omega}$
2	1/32	10	74	0.1346×10^{-4}	0.8964×10^{-6}
2	1/64	10	81	0.3370×10^{-5}	0.9051×10^{-6}
2	1/128	10	83	0.8435×10^{-6}	0.9625×10^{-6}
2	1/32	100	63	0.1347×10^{-4}	0.9112×10^{-6}
2	1/64	100	69	0.3371×10^{-5}	0.9263×10^{-6}
2	1/128	100	71	0.8443×10^{-6}	0.9520×10^{-6}
2.5	1/32	10	159	0.4112×10^{-4}	0.9483×10^{-6}
2.5	1/64	10	194	0.1029×10^{-4}	0.9956×10^{-6}
2.5	1/128	10	211	0.2577×10^{-5}	0.9705×10^{-6}
2.5	1/32	100	135	0.4112×10^{-4}	0.9733×10^{-6}
2.5	1/64	100	166	0.1029×10^{-4}	0.9624×10^{-6}
2.5	1/128	100	180	0.2577×10^{-5}	0.9609×10^{-6}
3	1/32	10	377	0.1027×10^{-3}	0.9992×10^{-6}
3	1/64	10	672	0.2569×10^{-4}	0.9967×10^{-6}
3	1/32	100	321	0.1027×10^{-3}	0.9818×10^{-6}
3	1/64	100	570	0.2569×10^{-4}	0.9991×10^{-6}

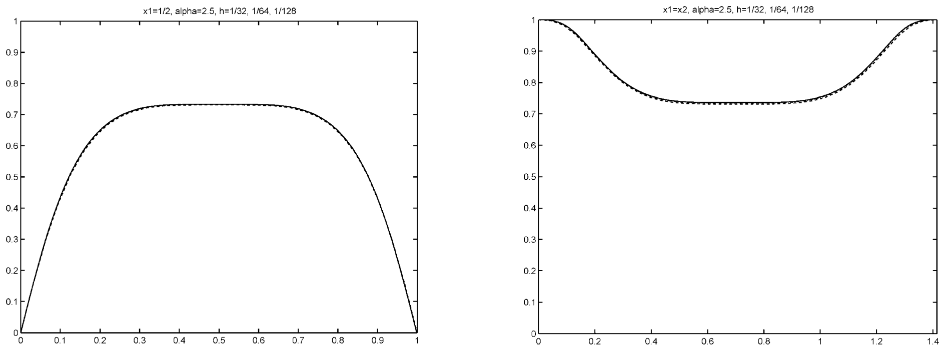
Table 2
Second test problem: summary of numerical results

α	h	τ	n_{it}	$\|D_h^2\psi_h^c - \mathbf{p}_h^c\|_{0,\Omega} / \|\mathbf{p}_h^c\|_{0,\Omega}$
2	1/32	10	67	0.9992×10^{-5}
2	1/64	10	70	0.9590×10^{-5}
2	1/128	10	75	0.9831×10^{-5}
2.5	1/32	10	158	0.9872×10^{-5}
2.5	1/64	10	167	0.9801×10^{-5}
2.5	1/128	10	168	0.9894×10^{-5}
3	1/32	10	978	0.9996×10^{-5}
3	1/64	10	1000	0.7865×10^{-4}
3	1/128	10	1000	0.8120×10^{-4}



(c)

Fig. 1. 2nd test problem: (a) ($\alpha = 1$, $h = 1/128$, $\tau = 10$); (b) ($\alpha = 2$, $h = 1/128$, $\tau = 10$); (c) ($\alpha = 3$, $h = 1/128$, $\tau = 10$).



(a) (b)

Fig. 2. Graph of ψ_h^c restricted to (a) $x_1 = 1/2$; (b) $x_1 = x_2$, ($\alpha = 2.5$, $h = 1/32$, $1/64$, $1/128$).

$\{\psi_h^c, \mathbf{p}_h^c\}$ denotes the computed approximation of $\{\psi, \mathbf{p}\}$. We took $\epsilon = 10^{-6}$. The results displayed in Table 1 call for several comments: (i) The larger τ, the faster the convergence of the iterative method, but the speed of convergence does not improve much as τ increases; similarly, the number of iterations necessary to achieve convergence does not depend much of h, for a given ϵ. (ii) For this test problem, we clearly have $\|\psi_h - \psi\|_{0,\Omega} = \mathrm{O}(h^2)$. (iii) The speed of convergence deteriorates as α increases; this is not surprising, since close to a solution of problem (2), the (Monge–Ampère) operator $\varphi \to \det D^2\varphi$ is a *nonlinear hyperbolic* one whose importance, relative to the operator $\varphi \to |\Delta\varphi|^2$, increases with α, making the problem more difficult to solve.

The *second family of test problems* corresponds to g defined by $g(x) = 0$ if $x \in \bigcup_{i=1}^4 \Gamma_i$, $g(x) = 1$ elsewhere on Γ, with $\Gamma_1 = \{x \mid x = \{x_1, x_2\},\ 1/4 < x_1 < 3/4,\ x_2 = 0\}$, $\Gamma_2 = \{x \mid x = \{x_1, x_2\},\ x_1 = 1,\ 1/4 < x_2 < 3/4\}$, $\Gamma_3 = \{x \mid x = \{x_1, x_2\},\ 1/4 < x_1 < 3/4,\ x_2 = 1\}$, and $\Gamma_4 = \{x \mid x = \{x_1, x_2\},\ x_1 = 0,\ 1/4 < x_2 < 3/4\}$. The above function $g \notin H^{3/2}(\Gamma)$ by far (actually, $g \notin H^{1/2}(\Gamma)$), implying that the corresponding (PE-D) problem has no solution in $H^2(\Omega)$. In order to overcome this difficulty we approximate g by g_δ defined as follows on the edge $\{x \mid x = \{x_1, x_2\},\ 0 \leqslant x_1 \leqslant 1,\ x_2 = 0\}$ of Ω: $g_\delta = 1$, if $0 \leqslant x_1 \leqslant 1/4 - \delta$ or $3/4 + \delta \leqslant x_1 \leqslant 1$, $g_\delta = 0$, if $1/4 + \delta \leqslant x_1 \leqslant 3/4 - \delta$, $g_\delta = \cos^2[1/4(x_1 - 1/4 + \delta)(\pi/\delta)]$ if $1/4 - \delta \leqslant x_1 \leqslant 1/4 + \delta$, $g_\delta = \cos^2[1/4(x_1 - 3/4 - \delta)(\pi/\delta)]$ if $3/4 - \delta \leqslant x_1 \leqslant 3/4 + \delta$, and similarly on the three other edges; above, δ is a 'small' positive parameter. The function g_δ is clearly in $H^{3/2}(\Gamma)$. Applying the methodology of the above sections leads — if $\delta = 1/16$ — to the results summarized in Table 2 and visualized in Figs. 1 and 2 (with —— , $- \cdot - \cdot -\cdot$, and $- - -$ corresponding to $h = 1/32, 1/64$, and $1/128$, respectively). The solution is clearly an increasing function of α and the convergence of ψ_h to a limit ψ as $h \to 0$ is clear from Fig. 2.

Acknowledgements

The authors thank L.A. Caffarelli for introducing them to the Pucci's equation and some of its fascinating properties. The support of NSF (Grant DMS-0209066) is also acknowledged.

References

[1] R.P. Brent, Algorithms for Minimization without Derivatives, Dover Publications, Mineola, NY, 2002.

[2] X. Cabré, Topics in regularity and qualitative properties of solutions of nonlinear elliptic equations, Discrete and Continuous Dynamical Systems 8 (2) (2002) 289–302.

[3] L.A. Caffarelli, X. Cabré, Fully Nonlinear Elliptic Equations, American Mathematical Society, Providence, RI, 1995.

[4] E.J. Dean, R. Glowinski, Numerical solution of the two-dimensional elliptic Monge–Ampère equation with Dirichlet boundary conditions: a least squares approach, C. R. Acad. Sci. Paris, Ser. I 339 (12) (2004) 887–892.

[5] E.J. Dean, R. Glowinski, Numerical solution of the two-dimensional elliptic Monge–Ampère equation with Dirichlet boundary conditions: an augmented Lagrangian approach, C. R. Acad. Sci. Paris, Ser. I 336 (2003) 779–784.

[6] E.J. Dean, R. Glowinski, Numerical methods for fully nonlinear elliptic equations of the Monge–Ampère type, Comput. Methods Appl. Mech. Engrg., in press.