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C. R. Acad. Sci. Paris, Ser. I 341 (2005) 283–286

COMPTES RENDUS MATHEMATIQUE

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Algebra/Algebraic Geometry

Schur finiteness and nilpotency

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Received 8 April 2005; accepted after revision 18 July 2005

Available online 25 August 2005

Presented by Pierre Deligne

Abstract

Let $\mathcal{A}$ be a $\mathbb{Q}$-linear pseudo-Abelian rigid tensor category. A notion of finiteness due to Kimura and (independently) O'Sullivan guarantees that the ideal of numerically trivial endomorphism of an object is nilpotent. We generalize this result to special Schur-finite objects. In particular, in the category of Chow motives, if X is a smooth projective variety which satisfies the homological sign conjecture, then Kimura-finiteness, a special Schur-finiteness, and the nilpotency of $CH^{ni}(X^i \times X^i)_{\mathrm{num}}$ for all i (where $n = \dim X$) are all equivalent. ***To cite this article: A. Del Padrone, C. Mazza, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Finitude de Schur et nilpotence. Soit $\mathcal{A}$ une catégorie tensorielle rigide pseudo-abélienne $\mathbb{Q}$-lineaire. Une notion de finitude de Kimura et (indépendamment) O'Sullivan garantit que l'idéal des endomorphismes numériquement triviaux d'un objet est nilpotent. Nous généralisons ce résultat à certains objets Schur-finis. En particulier, dans la catégorie des motifs de Chow, si X est une variété projective lisse purement de dimension n qui satisfait la conjecture homologique de signe, alors la finitude de Kimura, l'annulation du motif de X par un certain foncteur de Schur, et la nilpotence de $CH^{ni}(X^i \times X^i)_{\mathrm{num}}$ pour tous i, sont équivalentes. ***Pour citer cet article : A. Del Padrone, C. Mazza, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Finiteness conditions in tensor categories

Let $\mathcal{A}$ be a *pseudo-Abelian tensor category*, i.e., a '⊗-catégorie rigide sur F' as in [1, 2.2.2] in which idempotents split. We have F-linear *trace* maps $\mathrm{tr} : \mathrm{End}_{\mathcal{A}}(A) \to \mathrm{End}_{\mathcal{A}}(\mathbb{1})$ compatible with ⊗-functors, and F-submodules of *numerically trivial morphisms* $\mathcal{N}(A_1, A_2) := \{f \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \mid \mathrm{tr}(f \circ g) = 0, \text{ for all } g \in \mathrm{Hom}_{\mathcal{A}}(A_2, A_1)\}$.

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.07.010

We assume that $F = \mathrm{End}_{\mathcal{A}}(\mathbb{1})$ and it contains $\mathbb{Q}$. If F is a field, $\mathcal{N}$ is the biggest non-trivial $\otimes$-ideal of $\mathcal{A}$, and so it contains any morphism annihilated by some $\otimes$-functor.

Example 1 [1, Chapter 4]. Assume F is a field. For any admissible equivalence $\sim$ on algebraic cycles, motives of smooth projective varieties over a field k with coefficients in F form such a category $\mathcal{A} := \mathcal{M}_{\sim}(k)_F$. If X is a variety, we write $\mathfrak{h}(X)$ for its motive. For any $f \in \mathrm{End}_{\mathcal{A}}(\mathfrak{h}(X))$, $\mathrm{tr}(f) = \deg(\Gamma_f \cdot \Delta_X)$ and therefore $\mathcal{N}(\mathfrak{h}(X)) = \mathcal{Z}_{\sim}^{\dim(X)}(X \times X)_{F,\mathrm{num}}$ (numerically trivial correspondences of degree zero). If $\sim$ is finer than homological equivalence then any Weil cohomology H factors through a $\otimes$-functor on $\mathcal{A}$, and $\mathrm{tr}(f) = \sum_j (-1)^j \mathrm{Tr}(f|H^j(X))$ by the Lefschetz formula.

Recall that the partitions λ of an integer n give a complete set of mutually orthogonal central idempotents $d_\lambda := \frac{\dim V_\lambda}{n!} \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma)\sigma$ in the group algebra $\mathbb{Q}\Sigma_n$ (see [4]). We define an endofunctor on $\mathcal{A}$ by setting $S_\lambda(A) = d_\lambda(A^{\otimes n})$. This is a multiple of the classical Schur functor corresponding to λ. In particular, we define $\mathrm{Sym}^n(A) = S_{(n)}(A)$ and $\Lambda^n(A) = S_{(1^n)}(A)$. The following definitions are directly inspired by [3,7] (see [2,6,8] for further reference).

Definition 1.1. An object A of $\mathcal{A}$ is Schur-finite if there is a partition λ such that $S_\lambda(A) = 0$. If $S_\lambda(A) = 0$ with λ of the form (n) (respectively, $\lambda = (1^n)$) then A is called odd (respectively, even). We say that A is Kimura-finite if $A = A_+ \oplus A_-$ with A_+ even and A_- odd.

Every Kimura-finite object is Schur-finite, but the converse fails, for example, in the category of super-representations of $GL(p|q)$. In [7, 7.5] and [2, 9.1.14] it was proven that if A is a Kimura-finite object then the ideal $\mathcal{N}(A)$ is nilpotent.

In the case of Example 1, an interesting consequence of the nilpotence of $\mathcal{N}(M)$ is that a summand N of M is zero if and only if its cohomology is zero (the idempotent defining N must then be nilpotent). The nilpotency was used in [6, Theorem 7] to show the equivalence of Bloch's conjecture for a smooth projective surface X with $p_g = 0$ and the Kimura-finiteness of the motive of X, improving [7, 7.7].

Albeit in general Schur-finiteness is not sufficient to get the nilpotency of $\mathcal{N}(A)$ (see [2, 10.1.1]), we will identify additional conditions which imply the nilpotency. In the category of motives we will show that for a motive which is Kimura-finite modulo homological equivalence, the Kimura-finiteness modulo rational equivalence is equivalent to the Schur-finiteness for a particular rectangle.

2. A technical result

Theorem 2.1. *Suppose that $S_\lambda(A) = 0$ for a partition λ of $n \geqslant 2$ with a_λ rows and b_λ columns. Let $s := a_\lambda + b_\lambda - 1$ be the length of its biggest hook ν, and $r := n - s$. Assume that either λ is a hook or that there is a $g \in \mathrm{End}_{\mathcal{A}}(A)$ with trace $t := \mathrm{tr}(g) = \cdots = \mathrm{tr}(g^{\circ r})$, and $t \notin \{-(b_\lambda - 2), \ldots, a_\lambda - 2\}$. Then $f^{\circ(s-1)} = 0$ for each $f \in \mathcal{N}(A)$, and so $\mathcal{N}(A)$ is nilpotent.*

Proof. The last statement follows from [2, 7.2.8]: $\mathcal{N}(A)^{2^{s-1}-1} = 0$.

For $\sigma \in \Sigma_n$, we index the corresponding decomposition of $\{1, \ldots, n\}$ into disjoint cycles $\gamma_1, \ldots, \gamma_n$ so that the support of γ_1 contains 1; moreover we define l_i to be the order of the cycle γ_i, and $L = L(\sigma) := \max_i\{l_i\}$ to be the maximum length of the cycles of σ.

As $S_\lambda(A) = 0$ we have $\sum_\sigma \chi_\lambda(\sigma) \cdot \sigma \circ f_1 \otimes \cdots \otimes f_n = 0$ for any $f_1, \ldots, f_n \in \mathrm{End}_{\mathcal{A}}(A)$. By the Murnaghan–Nakayama rule (see [4, Problem 4.45]) $\chi_\lambda(\sigma) = 0$ if $L(\sigma) > s$. Hence [2, 7.2.6] with $A_1 = \cdots = A_n = A$, gives that in $\mathrm{End}_{\mathcal{A}}(A)$

$$\sum_{\sigma\in\Sigma_n:\ L(\sigma)\leqslant s} \chi_\lambda(\sigma)\cdot t_\sigma\cdot f_{\gamma_1}=0,$$

where $f_{\gamma_1}:=f_{\gamma_1^{l_1-1}(1)}\circ\cdots\circ f_{\gamma_1(1)}\circ f_1$, $t_\sigma:=\prod_{j=2}^q t_{\sigma,j}$, and $t_{\sigma,j}:=\mathrm{tr}(f_{\gamma_j{}^{l_j-1}(k_j)}\circ\cdots\circ f_{\gamma_j(k_j)}\circ f_{k_j})$ with k_j any element in the support of γ_j (if $l_1=n$, i.e. $q=1$, then $t_\sigma=1$).

Set $f_1:=\mathrm{Id}_A$ and $f_2=\cdots=f_s:=f$ (still no restrictions on $f_{s+1},\ldots,f_n$). If $\mathrm{Supp}(\gamma_1)\subsetneq\{1,\ldots,s\}$, not all of the f's are in the composition f_{γ_1}, hence at least one of them must appear in a trace $\mathrm{tr}(f_{\gamma_j{}^{l_j-1}(k_j)}\circ\cdots\circ f_{\gamma_j(k_j)}\circ f_{k_j})$. But f is numerically trivial, so $t_\sigma=0$ for any such σ, and

$$0=\sum_{\sigma\in\Sigma_n:\ \mathrm{Supp}(\gamma_1)=\{1,\ldots,s\}}\chi_\lambda(\sigma)\cdot t_\sigma\cdot f_{\gamma_1}=\Bigg(\sum_{\sigma\in\Sigma_n:\ \mathrm{Supp}(\gamma_1)=\{1,\ldots,s\}}\chi_\lambda(\sigma)\cdot t_\sigma\Bigg)f^{\circ(s-1)}=x\cdot f^{\circ(s-1)},$$

where $x:=\sum_{\sigma\in\Sigma_n:\ \mathrm{Supp}(\gamma_1)=\{1,\ldots,s\}}\chi_\lambda(\sigma)\cdot t_\sigma\in F$. It is enough to show $x\neq 0$ for some choice of the f_i's.

If $r=0$ then $\lambda=\nu=(n-j,1^j)$ is itself a hook, $t_\sigma=1$ for any σ with $l_1=n$ and by [4, Exercise 4.16] x is just $(n-1)!(-1)^j\neq 0$, hence $\mathcal{N}(A)$ is nilpotent.

If λ is not a hook let $\delta:=\lambda\setminus\nu$. The element $x\in F$ is a sum over $\sigma=\gamma_1\circ\sigma'$ such that γ_1 is an s-cycle of $\{1,\ldots,s\}$ and σ' is a permutation of $\{s+1,\ldots,n\}$, so by Murnaghan–Nakayama $\chi_\lambda(\sigma)=\chi_{\lambda\setminus\nu}(\sigma')$, and $x=(-1)^{a_\delta-1}|\{s-\text{cycles of }\Sigma_n\}|\sum_{\sigma'\in\Sigma_r}\chi_\delta(\sigma')\cdot t_\sigma$. Thus we are reduced to study elements of the form

$$y(\delta;g_1,\ldots,g_r):=\sum_{\sigma\in\Sigma_r}\chi_\delta(\sigma)\cdot\prod_{j=1}^q t_{\sigma,j},$$

where we can choose freely $g_1,\ldots,g_r\in\mathrm{End}_{\mathcal{A}}(A)$.

Take $g\in\mathrm{End}_{\mathcal{A}}(A)$ as in the hypothesis, then $y(\delta;g,\ldots,g)=\sum_{\sigma\in\Sigma_r}\chi_\delta(\sigma)\cdot t^{|\text{cycles of }\sigma|}$ is the polynomial in $t=\mathrm{tr}(g)$ called the *content polynomial* of δ. It decomposes as $y(\delta;g)=\chi_\delta(\mathrm{Id}_{\Sigma_r})\cdot\prod_{(i,j)\in\delta}(t+j-i)$, then $y(\delta;g)=0$ if and only if $\mathrm{tr}(g)\in\{-(b_\delta-1),\ldots,a_\delta-1\}\subseteq\{-(b_\lambda-2),\ldots,a_\lambda-2\}$. By hypothesis, there is a g such that $y(\delta;g)\neq 0$, which implies that $x\neq 0$, which in turn implies that f is nilpotent. Hence the theorem is proven. □

Remark 1 (*B. Kahn*). The existence of a $g\in\mathrm{End}_{\mathcal{A}}(A)$ with $\mathrm{tr}(g)\neq 0$ is not enough to ensure the nilpotency of $\mathcal{N}(A)$ with A Schur-finite. In [2, 10.1.1] it is exhibited a non-zero Schur-finite object A' with $\mathcal{N}(A')=\mathrm{End}_{\mathcal{A}}(A')$: it suffices to look at $A:=A'\oplus\mathbb{1}^n$.

Conjecture 2.2. *From numerical evidence* [9] *we conjecture a stronger version of Theorem* 2.1. *Let A be an object with two endomorphisms π_1 and π_2 such that $a:=\mathrm{tr}(\pi_1)=\mathrm{tr}(\pi_1^{\circ i})$ for all i, $b:=\mathrm{tr}(\pi_2)=\mathrm{tr}(\pi_2^{\circ j})$ for all j, and $\mathrm{tr}(\pi_1^{\circ i}\circ\pi_2^{\circ j})=0$ for all i and j. If $S_\lambda(A)=0$ where $\lambda\not\supset(b+2)^{a+2}$, then $y(\lambda\setminus\nu;\alpha_1\pi_1+\alpha_2\pi_2)\neq 0$* (*as a polynomial in α_1 and α_2*) *and hence $\mathcal{N}(A)$ is nilpotent.*

3. Motives and nilpotency

Let now $\mathcal{A}$ be the category of Chow motives $\mathcal{M}_{\mathrm{rat}}(k)_{\mathbb{Q}}$ (Example 1), let H be any Weil cohomology, and let X be a smooth projective variety. The cohomology $H(X)$ is a super vector space of dimension $(d_{\mathrm{ev}},d_{\mathrm{odd}})$, and we set $\lambda_{H(X)}:=((d_{\mathrm{odd}}+1)^{d_{\mathrm{ev}}+1})$ (the rectangle with $d_{\mathrm{odd}}+1$ columns and $d_{\mathrm{ev}}+1$ rows). By [3, 1.9], $S_\lambda(H(X))\neq 0$ if and only if $\lambda\not\supset\lambda_{H(X)}$. Hence, $S_\lambda(\mathfrak{h}(X))\neq 0$ if $\lambda\not\supset\lambda_{H(X)}$. So $S_\lambda(\mathfrak{h}(X))=0$ implies that $\lambda\supset\lambda_{H(X)}$.

Recall the 'homological sign conjecture' (due to Jannsen, see [1, 5.1.3]): we say that X satisfies the conjecture $C^+(X)$ if the projections on the even and the odd part of the cohomology are algebraic. This conjecture is stable under products, and it holds true, with respect to classical cohomologies, for Abelian varieties and smooth projective varieties of dimension at most two. It can be shown that $C^+(X)$ is equivalent to the Kimura-finiteness of the motive of X modulo homological equivalence.

Proposition 3.1. *Let X be a smooth projective variety, and let λ be a partition with at most $d_{\mathrm{ev}}+1$ rows or $d_{\mathrm{odd}}+1$ columns. If $S_\lambda(\mathfrak{h}(X))=0$ (and hence $\lambda\supset\lambda_{H(X)}$) and $C^+(X)$ holds, then $\mathcal{N}(\mathfrak{h}(X))$ is nilpotent. Moreover, if X is a surface with $p_g=0$, Bloch's conjecture holds for X.*

Proof. By $C^+(X)$ there are two cycles π_+ and π_- inducing the projections on the even and odd cohomology. Then $d_{\mathrm{ev}}=\mathrm{tr}(\pi_+)=\mathrm{tr}(\pi_+^{\circ i})$ for all i, and $-d_{\mathrm{odd}}=\mathrm{tr}(\pi_-)=\mathrm{tr}(\pi_-^{\circ j})$ for all j. Then either π_+ or π_- satisfies the condition of Theorem 2.1, and therefore $\mathcal{N}(\mathfrak{h}(X))$ is nilpotent. Bloch's conjecture is now a formal consequence of [7, 7.6 and 7.7]. □

Theorem 3.2. *Let X be a smooth projective variety. Under $C^+(X)$ the following are equivalent*:

1) $\mathfrak{h}(X)$ *is Kimura-finite*; 2) $S_{\lambda_{H(X)}}(\mathfrak{h}(X))=0$; 3) $\mathcal{N}(\mathfrak{h}(X^n))$ *is nilpotent for all* $n\geqslant 1$.

Proof. It is easy to show that $1\Rightarrow 2$. For $3\Rightarrow 1$ we proceed as follows. As $C^+(X)$ holds and $\mathcal{N}(\mathfrak{h}(X))$ is nilpotent, then there exist two motives X_+ and X_- whose cohomologies are exactly the even and the odd part of $H(X)$. It is now easy to prove that $\mathfrak{h}(X)=M_+\oplus M_-$ with M_+ even and M_- odd because it will be enough to check it in cohomology. We need to verify $2\Rightarrow 3$. Assume that $S_{\lambda_{H(X)}}(\mathfrak{h}(X))=0$. From the proof of [3, Corollary 1.13], we find that $S_{\lambda_{H(X^n)}}(\mathfrak{h}(X^n))=S_{\lambda_{H(X^n)}}(\mathfrak{h}(X)^{\otimes n})=0$. Since $C^+(X^n)$ holds true, Proposition 3.1 gives that $\mathcal{N}(\mathfrak{h}(X^n))$ is nilpotent. □

If Conjecture 2.2 is true, then Bloch's conjecture holds for any smooth projective surface X with $p_g=0$ such that $S_\lambda(\mathfrak{h}(X))=0$ for $\lambda\not\supset(d_{\mathrm{odd}}(X)+2)^{d_{\mathrm{ev}}(X)+2}$.

Acknowledgements

This article was inspired by the preprint [5] by V. Guletskiĭ. Both authors would like to thank the Institut de Mathématiques de Jussieu for hospitality while part of this manuscript was written. For financial support, the first author would like to thank the Fondazione Carige and the second author would like to thank Sergio Serapioni (Honorary President of the Società Trentina Lieviti, Trento, Italy).

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