



Differential Geometry

Concentration of the first eigenfunction  
for a second order elliptic operator

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**Abstract**

We study the semi-classical limits of the first eigenfunction of a positive second order operator on a compact Riemannian manifold when the diffusion constant  $\epsilon$  goes to zero. We assume that the first order term is given by a vector field  $b$ , whose recurrent components are either hyperbolic points or cycles or two dimensional torii. The limits of the normalized eigenfunctions concentrate on the recurrent sets of maximal dimension where the topological pressure [Y. Kifer, Principal eigenvalues, topological pressure and stochastic stability of equilibrium states, *Israel J. Math.* 70 (1990) (1) 1–47] is attained. On the cycles and torii, the limit measures are absolutely continuous with respect to the invariant probability measure on these sets. We have determined these limit measures, using a blow-up analysis. *To cite this article: D. Holcman, I. Kupka, C. R. Acad. Sci. Paris, Ser. I 341 (2005).*

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**Résumé**

**Concentration de la première fonction propre pour un opérateur elliptique du second ordre.** Nous étudions sur une variété Riemannienne compacte, les limites semiclassiques de la première fonction propre associée à un opérateur positif du second ordre positif divers quand la constante de diffusion  $\epsilon$  tend vers zéro. Nous supposons que le terme d'ordre un est un champ de vecteur  $b$ , dont les ensembles récurrents sont des points hyperboliques ou des cycles ou des tores à deux dimensions. Les limites de la fonction propre normalisée sont concentrées sur les ensembles récurrents de dimension maximale où la pression topologique est atteinte. Sur les cycles et les tores, les mesures limites sont absolument continues par rapport à la mesure de probabilité invariante par  $b$ . Nous avons déterminé ces limites en utilisant une analyse de type blow-up. *Pour citer cet article : D. Holcman, I. Kupka, C. R. Acad. Sci. Paris, Ser. I 341 (2005).*

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## 1. Statement of the problem

On a compact Riemannian manifold  $(M, g)$ , we study the semi-classical limits of the first eigenfunction of a positive second order operator

$$L_\epsilon = \epsilon \Delta_g + \theta(b) + c, \quad (1)$$

when the diffusion constant  $\epsilon$  goes to zero.  $\Delta_g$  is the Laplace–Beltrami operator and  $\theta(b)$  is the Lie derivative in the direction  $b \in TM$ . The function  $c$  is chosen so that  $L_\epsilon$  is positive. Note that  $L_\epsilon$  cannot be conjugated in general to a self-adjoint operator by scalar multiplication. The Krein–Rutman theorem implies that the first eigenvalue  $\lambda_\epsilon$  is real positive, simple and the associated eigenfunction  $u_\epsilon$  is of a fixed sign (see [6,7]). The  $L_2$  normalized positive eigenfunction  $u_\epsilon$  ( $\int_{V_n} u_\epsilon^2 dV_g = 1$ ), is solution of

$$\epsilon \Delta_g u_\epsilon + (b, \nabla u_\epsilon) + c u_\epsilon = \lambda_\epsilon u_\epsilon.$$

Let us introduce the following notation,

$$\Omega = b - \nabla \mathcal{L}, \quad v_\epsilon = e^{-\mathcal{L}/(2\epsilon)} u_\epsilon, \quad c_\epsilon = \epsilon \left( c + \frac{\Delta_g \mathcal{L}}{2} \right) + \Psi_{\mathcal{L}}, \quad \Psi_{\mathcal{L}} = \frac{1}{4} (\|\nabla \mathcal{L}\|^2 + 2(\nabla \mathcal{L}, \Omega)),$$

where the function  $\mathcal{L}: M \rightarrow \mathbb{R}$  belongs to a special class of Lyapunov functions, associated to the vector field  $b$ , as described in [5]. Eq. (2) is transformed into  $L_\epsilon(v_\epsilon) = \epsilon^2 \Delta_g v_\epsilon + \epsilon(\Omega, \nabla v_\epsilon) + c_\epsilon v_\epsilon = \epsilon \lambda_\epsilon v_\epsilon$ , and we impose the normalization condition  $\int_{V_n} v_\epsilon^2 dV_g = 1$ .

In this situation, a fundamental problem is to study (1) the limits of  $\lambda_\epsilon$  (2) the limits of the measures  $v_\epsilon^2 dV_g$ , as  $\epsilon$  goes to zero.

### 1.1. History of the problem

The study of  $\lambda_\epsilon$  has its origin in the work of Kolmogorov about the mean first passage time (MFPT) of a random particle to the boundary of a bounded domain. For domains in  $\mathbb{R}^n$ , very interesting contributions to the first problem were made, in particular by Freidlin and Wentzell [4] and Devinatz and Friedman [1,3]. In the global situation (compact Riemannian manifold), Kifer in [6] made a fundamental contribution to the study of the limit of  $\lambda_\epsilon$ : When the recurrent set of the drift  $b$  is a finite union of hyperbolic components [8],  $\lambda_\epsilon$  has a unique limit, which can be explicitly determined in terms of the topological pressure of the components.

### 1.2. Limits of the first eigenfunction

Problem (2) was much less studied than problem (1). However in the local situation, there are some interesting contributions in the paper [2] by Devinatz and Friedman. Otherwise very little was known. Here, we study this problem (2) in the spirit of Kifer. We consider drift  $b$  such that the recurrent set is the union of a finite number of hyperbolic components, which are either points, cycles or 2-dimensional torii.

Let us mention that problem (2) is related to questions studied in the unique ergodic conjecture [9], where the limits of eigenfunctions are studied.

## 2. Notation and assumptions

We consider here vector fields  $b$ , related to the Morse–Smale fields, which satisfy some strong compatibility conditions with respect to the metric  $g$ . More specifically we shall assume that for each field  $b$ :

- (I) The recurrent set is a finite union of stationary points, limit cycles and two-dimensional torii.

- (II) The stationary points are hyperbolic and for each such  $P$  the stable and unstable manifolds belonging to  $P$  are orthogonal at  $P$  with respect to the metric  $g$ .
- (III) Any limit cycle  $S$  has a tubular neighborhood  $T_S$  provided with a covering map  $\Phi : \mathbb{R}^{m-1} \times \mathbb{R} \rightarrow T_S$  having the following properties:
  - (a) for all  $(x', \theta) \in \mathbb{R}^{m-1} \times \mathbb{R}$ ,  $\Phi^{-1} \circ \Phi(x', \theta) = \{(x', \theta + nT_S) \mid n \in \mathbb{Z}\}$ ,  $T_S$  minimal period of  $S$ ;
  - (b) at any point  $(0, \theta) \in \mathbb{R}^{m-1} \times \mathbb{R}$ ,  $(\Phi)^*g_{(0,\theta)} = \sum_{n=1}^{m-1} dx_n^2 + g^{mm}(\theta) d\theta^2$  ( $\theta = x^m$ );
  - (c) at any point  $(0, \theta) \in \mathbb{R}^{m-1} \times \mathbb{R}$ ,  $(\Phi)_*b = \frac{\partial}{\partial \theta} + \sum_{i,j=1}^{m-1} B_{ij}x_j \frac{\partial}{\partial x_i}$ , up to term of order two in  $x' = (x_1, \dots, x_{m-1})$ , canonical coordinates on  $\mathbb{R}^{m-1}$ .
  - (d) the  $(m-1) \times (m-1)$  matrix  $B = \{B_{ij} \mid 1 \leq i, j \leq m-1\}$  is hyperbolic and its stable and instable spaces are orthogonal with respect to the Euclidean metric  $\sum_{n=1}^{m-1} x_n^2$  on  $\mathbb{R}^{m-1}$ .
- (IV) Any 2-dimensional torus  $R$  has a tubular neighborhood  $T_R$  provided with a diffeomorphism  $\Phi : \mathbb{R}^{m-2} \times \mathbb{R}^2 \rightarrow T_R$  having the following properties:
  - (a) at any point  $(0, \theta) \in \mathbb{R}^{m-2} \times \mathbb{R}^2$ ,  $\theta = (\theta_1, \theta_2)$ ,  $\theta_1, \theta_2$  cyclic coordinates,

$$(\Phi^{-1})^*g_{(0,\theta)} = \sum_{n=1}^{m-2} dx_n^2 + a(\theta) d\theta_1^2 + 2b(\theta) d\theta_1 d\theta_2 + c(\theta) d\theta_2^2;$$

- (b) at any point  $(0, \theta) \in \mathbb{R}^{m-2} \times \mathbb{R}^2$ ,  $(\Phi)_*(b)(0, \theta) = k_1 \frac{\partial}{\partial \theta_1} + k_2 \frac{\partial}{\partial \theta_2} + \sum_{i,j=1}^{m-2} B_{ij}x_j \frac{\partial}{\partial x_i}$  where:
  - (i) the  $(m-2) \times (m-2)$  matrix  $B = \{B_{ij} \mid 1 \leq i, j \leq m-2\}$  is hyperbolic and its stable and instable spaces are orthogonal with respect to the Euclidean metric  $\sum_{n=1}^{m-2} x_n^2$  on  $\mathbb{R}^{m-2}$ ,
  - (ii)  $k_1, k_2 \in \mathbb{R}$  and  $\frac{k_1}{k_2} \in \mathbb{R} - \mathbb{Q}$ . A torus with such a flow will be called an irrational torus.
  - (iii) Small divisor condition: There exist constants  $C > 0, \alpha > 0$  such that for all  $m_1, m_2 \in \mathbb{Z}$ ,  $|m_1k_1 + m_2k_2| \geq C(m_1^2 + m_2^2)^\alpha$ .

### 3. Statement of the main result

The spirit of our result can be summarized as follow:

“On a Riemannian manifold, for any choice of a special Lyapunov function  $\mathcal{L}$ , vanishing at order 2 on the recurrent sets of the field, the limits as  $\epsilon$  tends to 0 of the normalized measures  $e^{-\mathcal{L}/(2\epsilon)} u_\epsilon^2 dV_g$  are concentrated on the components of the recurrent sets which are of maximal dimension and where the topological pressure is achieved.”

The construction of the function  $\mathcal{L}$  is detailed in [5].

**Theorem 3.1.** *On a compact Riemannian manifold  $(M, g)$ , let  $b$  be a Morse–Smale vector field and  $\mathcal{L}$  be a special Lyapunov function for  $b$ . Consider the normalized positive eigenfunction  $u_\epsilon > 0$  of the operator  $L_\epsilon = \epsilon \Delta + \theta(b) + c$ , associated to the first eigenvalue  $\lambda_\epsilon$ .*

- (i) *The recurrent set  $S$  of  $b$  is a union of a finite set of stationary points  $S^s$ , a finite set of periodic orbits  $S^P$  and a finite set of two-dimensional irrational torii  $S^t$ . The limit set of a normalized measure  $\frac{u_\epsilon^2 e^{-\mathcal{L}/\epsilon} dV_g}{\int_{V_n} u_\epsilon^2 e^{-\mathcal{L}/\epsilon} dV_g}$ , is contained in the set of probability measure  $\mu$  of the form*

$$\mu = \sum_{P \in S_{ip}^s} c_P \delta_P + \sum_{\Gamma \in S_{ip}^P} a_\Gamma \delta_\Gamma + \sum_{\mathbb{T} \in S_{ip}^t} b_{\mathbb{T}} \delta_{\mathbb{T}}$$

where  $S_{ip}^s$  (resp.  $S_{ip}^P, S_{ip}^t$ ) is the subset of  $S^s$  (resp.  $S^P, S^t$ ), where the topological pressure is attained.  $\delta_P$  is the Dirac measure at  $P$ .

For  $\Gamma \in S^p$  and  $h \in C(V)$ ,  $\delta_\Gamma(h) = \int_0^{T_\Gamma} f_\Gamma(\theta) h(\Gamma(\theta)) d\theta$ , where  $\theta \in \mathbb{R} \rightarrow \Gamma(\theta) \in V$  is a solution of  $b$  representing  $\Gamma$  (see the notations for the precise definition of  $\theta$ ). The periodic function  $f_\Gamma$  is given by

$$f_\Gamma(\theta) = \exp \left\{ - \int_0^\theta c(\Gamma(s)) ds + \frac{\theta}{T_\Gamma} \int_0^{T_\Gamma} c(\Gamma(s)) ds \right\}$$

and  $T_\Gamma$  is the minimal period of  $\Gamma$ .

For  $\mathbb{T} \in S^t$  and  $h \in C(V)$ ,  $\delta_{\mathbb{T}}(h) = \int_{\mathbb{T}} h(\theta_1, \theta_2) f_{\mathbb{T}}(\theta_1, \theta_2) dS_{\mathbb{T}}$ , where  $dS_{\mathbb{T}}$  is the unique probability measure on  $\mathbb{T}$  invariant under the action of the field  $b$  and  $f_{\mathbb{T}}$  is the unique solution of maximum 1, of the equation

$$k_1 \frac{\partial f}{\partial \theta_1} + k_2 \frac{\partial f}{\partial \theta_2} + cf = \mu_2 f \quad \text{where } \mu_2 = \int_{\mathbb{T}} c dS_{\mathbb{T}}.$$

- (ii) The coefficients  $c_P, a_\Gamma, b_{\mathbb{T}}$  obey the following rule: If at least one coefficient  $b_{\mathbb{T}} > 0$ , then for all cycles  $a_\Gamma = 0$  and all points  $c_P = 0$ . If all coefficients  $b_{\mathbb{T}} = 0$ , and at least one coefficient  $a_\Gamma > 0$  then all  $c_P = 0$ .

#### 4. Remark on the proofs and future perspectives

The basic techniques are the Blow-up and stochastic analysis and the theory of Ornstein–Uhlenbeck operator, but they are too involved to be presented here. As a buy product, we estimate the decay of the first eigenfunction near the recurrent sets.

We conjecture that under  $L_p$  normalization, where  $p > 1$ , the limit measures are similar to the one we obtained, except that the coefficients are different. To our knowledge, it is still unknown if the limit measure is unique or not. But this is a difficult problem as explained partially in [10] and in more details in [5].

#### References

- [1] A. Devinatz, R. Ellis, A. Friedman, The asymptotic behavior of the first real eigenvalue of second order elliptic operators with a small parameter in the highest derivatives. II, Indiana Univ. Math. J. 23 (1973–1974) 991–1011.
- [2] A. Devinatz, A. Friedman, Asymptotic behavior of the principal eigenfunction for a singularly perturbed Dirichlet problem, Indiana Univ. Math. J. 27 (1) (1978) 143–157.
- [3] A. Friedman, The asymptotic behavior of the first real eigenvalue of a second order elliptic operator with a small parameter in the highest derivatives, Indiana Univ. Math. J. 22 (1972–1973) 1005–1015.
- [4] M.I. Freidlin, A.D. Wentzell, Random Perturbations of Dynamical Systems, Grundlehren Math. Wiss., vol. 260, Springer-Verlag, New York, 1984.
- [5] D. Holcman, I. Kupka, Singular perturbation for the first eigenfunction and blow up analysis, Forum Math., May 2006, in press and <http://arxiv.org/pdf/math-ph/0412088>.
- [6] Y. Kifer, Principal eigenvalues, topological pressure and stochastic stability of equilibrium states, Israel J. Math. 70 (1) (1990) 1–47.
- [7] R. Pinsky, Positive Harmonic Functions and Diffusion, Cambridge Stud. Adv. Math., vol. 45, Cambridge University Press, Cambridge, 1995.
- [8] C. Robinson, Dynamical Systems. Stability, Symbolic Dynamics, and Chaos, second ed., CRC Press, 1995.
- [9] P. Sarnak, Arithmetic quantum chaos, First annual R.A. Blyth Lectures, 15–19 March 1993, University of Toronto, Department of Mathematics.
- [10] B. Simon, Semi-classical analysis of low lying eigenvalues. I. Nondegenerate minima: asymptotic expansions, Ann. Inst. H. Poincaré Sect. A (N.S.) 38 (3) (1983) 295–308.