Available online at www.sciencedirect.com

SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 341 (2005) 107–111

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Differential Geometry/Dynamical Systems

On Ishii's equation

Petre Birtea, Mircea Puta

West University of Timişoara, Blvd. V. Parvan 4, Timişoara 300223, Timis, Romania

Received 26 May 2005; accepted 31 May 2005

Available online 29 June 2005

Presented by Charles-Michel Marle

Abstract

We will study the dynamics of Ishii's equation using its Hamilton–Poisson formulation. ***To cite this article: P. Birtea, M. Puta, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Sur l'équation d'Ishii. On va étudier la dynamique de l'équation de Ishii en utilisant une réalisation Hamilton–Poisson de cette équation. ***Pour citer cet article : P. Birtea, M. Puta, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

L'équation de Ishii peut s'écrire sous la forme :

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = x_1 x_2. \end{cases}$$

Dans cette Note nous étudions sa géometrie Poisson et quelques aspects de sa dynamique.

1. Introduction

The third order Ishii's equation, see [2], has the following form:

$$\dddot{x} = \dot{x} x. \tag{1}$$

E-mail addresses: birtea@math.uvt.ro (P. Birtea), puta@math.uvt.ro (M. Puta).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.06.010

Written as a system it can be expressed:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = x_1 x_2. \end{cases} \tag{2}$$

Proposition 1.1. *For the system* (2) *the quantities*

$$H(x_1, x_2, x_3) = x_1 x_3 - \frac{1}{2} x_2^2 - \frac{1}{3} x_1^3 \tag{3}$$

and

$$C(x_1, x_2, x_3) = x_3 - \frac{1}{2} x_1^2 \tag{4}$$

are constants of motion.

These two constants of motion have a geometrical meaning, precisely:

Theorem 1.2. *The system* (2) *has a Hamilton–Poisson realization with the Hamiltonian H given by* (3) *and with the Poisson structure given by*:

$$\{f, g\} = x_1 \left(\frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_3} - \frac{\partial f}{\partial x_3} \frac{\partial g}{\partial x_2} \right) + \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2}.$$

Proof. One readily checks that

$$\dot{x}_i = \{x_i, H\}, \quad i = 1, 2, 3,$$

give the result. □

Remark 1. It is not hard to see that the function C given by (4) is a Casimir of our configuration $(\mathbb{R}^3, \{\cdot, \cdot\})$.

Theorem 1.3. *The system* (2) *has an infinite number of Hamilton–Poisson realizations.*

Proof. It is easy to see that

$$\left(\mathbb{R}^3, \{\cdot, \cdot\}_{\alpha,\beta}, H_{\delta,\gamma}\right),$$

where

$$\begin{aligned} &C_{\alpha\beta}(x_1, x_2, x_3) = \alpha \left(x_3 - \frac{1}{2} x_1^2 \right) + \beta \left(x_1 x_3 - \frac{1}{2} x_2^2 - \frac{1}{3} x_1^3 \right), \\ &\{f, g\}_{\alpha\beta} = -\nabla C_{\alpha\beta} (\nabla f \times \nabla g), \\ &H_{\delta\gamma}(x_1, x_2, x_3) = \delta \left(x_3 - \frac{1}{2} x_1^2 \right) + \gamma \left(x_1 x_3 - \frac{1}{2} x_2^2 - \frac{1}{3} x_1^3 \right), \\ &\alpha, \beta, \gamma, \delta \in \mathbb{R}, \quad \alpha\gamma - \beta\delta = 1, \end{aligned}$$

are Hamilton–Poisson realization of the dynamics (2). □

Thus, the equations of motion for our system (2) are unchanged (so the trajectories of the motion in $\mathbb{R}^3$ remain unchanged) when the energy H and the Casimir C are replaced by $\mathrm{SL}(2, \mathbb{R})$ linear combinations of H and C.

2. Stability problem

It is easy to see that the equilibrium states of our system are

$$e_M = (M, 0, 0), \quad M \in \mathbb{R}.$$

Let A be the matrix of the linear part of our system, i.e.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & M & 0 \end{bmatrix}$$

and then the characteristic polynomial is

$$p_{A(e_M)}(x) = -x(x^2 - M). \tag{5}$$

Proposition 2.1. *The equilibrium states e_M have the following behavior*:

(i) e_M, $M > 0$ *are unstable*;
(ii) e_M, $M = 0$ *is unstable*;
(iii) e_M, $M < 0$ *are spectrally stable.*

Proof. The statement (i) and (iii) are consequences of the expression of the characteristic polynomial (5).

The statement (ii) follows if we take in account that the minimal polynomial of the matrix $A(e_0)$ is

$$m_{A(e_0)}(x) = x^3. \qquad \square$$

Next we will study the nonlinear stability of e_M for $M < 0$. In a neighborhood V_{e_M} of e_M we have $\nabla C(x) \neq 0$ and consequently we obtain a 2-dimensional integrable distribution in this neighborhood. Taking V_{e_M} sufficiently small, the Frobenius theorem guarantees that V_{e_M} is diffeomorphic with $(V_{e_M} \cap C^{-1}(C(e_M)) \times (a, b)$, where (a, b) is an open interval of $\mathbb{R}$ that contains $C(e_M) \neq 0$; see [4] for details about integrable distributions and the Frobenius theorem.

Our vector field it is mapped by this diffeomorphism into $(X|_{C^{-1}(C(e_M))}, 0)$, since C is a constant of motion.

The stability problem for e_M, $M < 0$ reduces to the study of stability of e_M for the vector field (2) restricted to the fiber $C^{-1}(C(e_M))$.

The following transformation

$$\begin{cases} x_1 = M + r\cos\theta, \\ x_2 = z, \\ x_3 = Mr\cos\theta - \dfrac{r^2\sin^2\theta}{2} + \dfrac{M^2}{2} \end{cases}$$

gives us a set of adapted coordinates for the fibration

$$V_{e_M} \cong \big(V_{e_M} \cap C^{-1}\big(C(e_M)\big)\big) \times (a, b),$$

where e_M is transformed into $(\frac{\pi}{2}, 0, 0)$ and (θ, z) are local coordinates on the fiber $V_{e_M} \cap C^{-1}(C(e_M))$.

With these new coordinates our system becomes

$$\begin{cases} \dot\theta = -\dfrac{2z}{r\sin\theta}, \\ \dot r = 0, \\ \dot z = Mr\cos\theta - \dfrac{r^2\sin^2\theta}{2} + \dfrac{M^2}{2}. \end{cases}$$

The function

$$H(\theta,z)=-\frac{M^3\sin^2\theta}{2}-\frac{M^3\cos\theta(2+\sin^2\theta)}{6}+\frac{M^3\cos\theta}{2}+\frac{M^3}{6}-\frac{z^2}{2}$$

is a constant of motion for $X_{|_{C^{-1}(C(e_M))}}$.

We have that $\nabla H((\frac{\pi}{2},0))=0$ and (Hess $H)((\frac{\pi}{2},0))<0$ which proves that $(\frac{\pi}{2},0)$ is a nonlinear stable equilibrium for $X_{|_{C^{-1}(C(e_M))}}$ and consequently e_M is nonlinear stable critical point of (2) for $M<0$.

In conclusion we have proved the following:

Theorem 2.2. *The equilibrium states e_M have the following behavior*:

(i) e_M, $M\geqslant 0$ *are unstable*;
(ii) e_M, $M<0$ *are stable.*

3. Lax formulation

We shall now discuss the Lax formulation along the trajectories of our dynamics (2). The notion of Lax formulation along the trajectories can be found in [1].

Consider the matrix $S=\operatorname{diag}(H,C,\lambda)$ where λ is an arbitrary parameter and $M=M(x_1,x_2,x_3)\in \mathrm{SL}(3,\mathbb{R})$, for each $(x_1,x_2,x_3)\in\mathbb{R}^3$. Then we have:

Theorem 3.1. *The dynamics* (2) *has the following Lax formulation*:

$$\dot A=[A,B],$$

where

$$A=MSM^{-1}$$

and

$$B=-\dot M M^{-1}.$$

Proof. By construction, we have $\dot S=0$. We must check that (A,B) is a Lax pair

$$\begin{aligned}\frac{\mathrm d}{\mathrm dt}A&=\dot MSM^{-1}+M\dot SM^{-1}-MSM^{-1}\dot MM^{-1}\\&=\dot MSM^{-1}-MSM^{-1}\dot MM^{-1}\\&=\dot MM^{-1}MSM^{-1}-MSM^{-1}\dot MM^{-1}\\&=[MSM^{-1},-\dot MM^{-1}]\\&=[A,B].\qquad\Box\end{aligned}$$

4. Integrability via a Weierstrass function

We shall prove now that the system (2) can be explicitly integrated via a Weierstrass function. To begin, let us observe that the relations:

$$x_3-\frac{1}{2}x_1^2=C\ \text{(constant)}$$

and

$$x_1x_3 - \frac{1}{2}x_2^2 - \frac{1}{3}x_1^3 = H \text{ (constant)}$$

lead us to:

$$x_2^2 = \frac{1}{3}x_1^3 + 2Cx_1 - 2H.$$

Since

$$(\dot{x}_1)^2 = x_2^2$$

we can conclude that:

$$(\dot{x}_1)^2 = \frac{1}{3}x_1^3 + 2Cx_1 - 2H. \tag{6}$$

If we take now:

$$x_1 = 12\mathcal{P}$$

our relation (6) becomes:

$$\dot{\mathcal{P}}^2 = 4\mathcal{P}^3 + \frac{C}{6}\mathcal{P} - \frac{H}{72} \tag{7}$$

and consequently we have proved, see [3] for details concerning Weierstrass function:

Theorem 4.1. *The dynamics* (6) *may be explicitly integrated via a Weierstrass function.*

References

[1] A. Goriely, Integrability and Nonintegrability of Dynamical System, Adv. Ser. Nonlinear Dynam., vol. 19, World Scientific, 2001.
[2] M. Ishii, Painleve property and algebraic integrability of single variable ordinary differential equations with dominants, Progr. Theor. Phys. 84 (1990) 386–391.
[3] D.F. Lawden, Elliptic Functions and Applications, Appl. Math. Sci., vol. 80, Springer, 1989.
[4] P. Libermann, C.-M. Marle, Symplectic Geometry and Analytical Mechanics, Reidel, 1987.