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## Numerical Analysis

# Approximation properties of lowest-order hexahedral Raviart–Thomas finite elements

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## Abstract

Basic interpolation results are settled for lowest-order hexahedral Raviart–Thomas finite elements. Convergence in  $H(\text{div})$  is proved for regular families of asymptotically parallelepiped meshes. The need of the asymptotically parallelepiped assumption is demonstrated with a numerical example. *To cite this article: A. Bermúdez et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).*  
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## Résumé

**Propriétés d'approximation des éléments finis de Raviart–Thomas hexaédriques d'ordre le plus bas.** Nous démontrons quelques résultats d'interpolation pour les éléments finis de Raviart–Thomas hexaédriques d'ordre le plus bas. Nous prouvons convergence dans l'espace  $H(\text{div})$  pour des familles régulières de maillages dont les éléments sont, asymptotiquement, des parallélépipèdes. La nécessité de cette hypothèse est montrée numériquement avec un exemple. *Pour citer cet article : A. Bermúdez et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).*  
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## Version française abrégée

Le but de cet Note est d'étudier les propriétés d'approximation des éléments finis de Raviart–Thomas hexaédriques d'ordre le plus bas. Ces éléments ont été utilisés avec succès dans beaucoup d'applications comme par exemple la vibroacoustique (voir [5,9,10]).

L'optimalité de l'approximation des fonctions régulières pour des maillages rectangulaires (dans 2D) ou cubiques (dans 3D) a été démontrée dans [14,15] et [13], respectivement. Pour des maillages quadrilatéraux plus

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généraux, l'optimalité en norme  $L^2$  est prouvée dans [15], mais la convergence optimale dans  $H(\text{div})$  n'y est démontrée que pour des familles de maillages dont les éléments soient asymptotiquement des parallélogrammes, plus précisément, formés de quadrilatères tels que la distance de leurs sommets aux sommets d'un parallélogramme soit d'ordre supérieur. Plus récemment, Arnold et al. [1,3] ont démontré qu'en effet l'ordre de convergence peut se détériorer quand on utilise des éléments finis non standard sur certains maillages quadrilatéraux (donc en 2D), même s'ils sont réguliers. Ces auteurs ont aussi démontré dans [2], la dégradation de la convergence pour des maillages de trapèzes par rapport à des maillages rectangulaires, dans le cas d'espaces d'éléments finis vectoriels définis par la transformée de Piola (dont un cas particulier sont les éléments de Raviart–Thomas).

Dans cet article nous montrons un résultat d'approximation (Théorème 3.1) pour des maillages d'hexaèdres quelconques, ce qui entraîne que la convergence dans  $H(\text{div})$  est préservée sur des maillages réguliers formés d'hexaèdres qui soient asymptotiquement des parallélépipèdes (Corollaire 3.2). Nous montrons aussi la nécessité de cette hypothèse avec un exemple numérique.

## 1. Introduction

The aim of this Note is to study the approximation properties of lowest-order hexahedral *Raviart–Thomas finite elements*. This kind of elements have been introduced by Raviart and Thomas in [14,15] for 2D meshes built on triangles or quadrilaterals and extended by Nédélec in [13] to 3D meshes built on tetrahedra or cubes. These elements have been successfully used in several many applications; for instance, to discretize fluid displacements in fluid-structure vibroacoustic interaction problems (v.g., [5,9,10]). In this type of problems, Raviart–Thomas elements based on hexahedra proved to be much more efficient than those based on tetrahedra (see [7,8]).

Raviart–Thomas elements are  $H(\text{div})$ -conforming. They provide optimal-order approximations in  $H(\text{div})$  of smooth vector fields on 2D shape-regular rectangular meshes ([14]) or 3D cubic meshes ([13]). A deep analysis of these elements in 2D has been made by Thomas in [15], where optimal-order  $L^2$ -convergence has been proved for general shape-regular quadrilaterals. However,  $H(\text{div})$ -convergence has been proved in the same reference only for *asymptotically parallelogram* meshes (i.e., meshes built on shape-regular quadrilaterals such that the distance of the vertices of each element to those of a parallelogram are higher order terms). To the best of the authors knowledge, a similar analysis has not been made on general hexahedra 3D meshes.

On the other hand, Arnold et al. [1,3] have recently noted, for various types of 2D finite elements, that the extension to general quadrilaterals of approximation results valid for rectangular elements is less straightforward than what it was typically thought. Moreover, they have shown that the order of convergence can deteriorate when some non-standard finite elements are used in distorted quadrilaterals, even though they are shape-regular. The same authors have considered in [2] the approximation properties of 2D quadrilateral finite element spaces of vector fields defined by the *Piola transform*, as it is the case of Raviart–Thomas elements. For these elements, they have shown degradation of the convergence order for some trapezoid meshes as compared to rectangular ones.

We show in this paper that  $H(\text{div})$ -convergence is preserved on shape-regular *asymptotically parallelepiped* 3D meshes and we demonstrate the need of this assumption by means of a numerical example.

## 2. The lowest-order Raviart–Thomas hexahedron

A *hexahedron* is the image of an invertible trilinear map defined on a reference element: the unit cube  $\widehat{K} := [0, 1]^3$ . (Throughout the paper, we use ‘hatted’ variables to denote magnitudes in the reference element.) Indeed, let  $\mathbf{F}_K : \widehat{K} \rightarrow \mathbb{R}^3$  be a trilinear one-to-one map with a strictly positive Jacobian:  $J_K(\hat{\mathbf{x}}) := \det[\widehat{D}\mathbf{F}_K(\hat{\mathbf{x}})] > 0 \forall \hat{\mathbf{x}} \in \widehat{K}$ , where  $\widehat{D}\mathbf{F}_K$  denotes the *Jacobian matrix* of the mapping  $\mathbf{F}_K$ . The corresponding hexahedron is  $K := \mathbf{F}_K(\widehat{K})$ . Notice that  $K$  does not need to have planar faces; the edges are always straight segments, but each face is planar if and only if its four vertices lie on a plane.

Let  $h_K$  be the *diameter* of  $K$ . Let  $\bar{\sigma}_K$  and  $\underline{\sigma}_K$  be two strictly positive constants such that

$$h_K^{-1} |\widehat{D}\mathbf{F}_K(\hat{\mathbf{x}})| \leq \bar{\sigma}_K \quad \forall \hat{\mathbf{x}} \in \widehat{K} \quad \text{and} \quad h_K^{-1} |D(\mathbf{F}_K^{-1})(\mathbf{x})|^{-1} \geq \underline{\sigma}_K \quad \forall \mathbf{x} \in K.$$

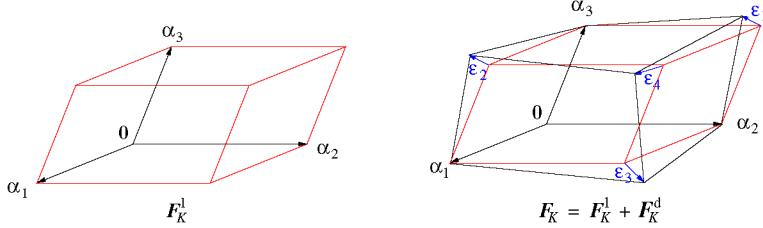


Fig. 1. Image of  $\widehat{K}$  by  $F_K^l$  (left) and  $F_K$  decomposed as  $F_K^l + F_K^d$  (right).

Here and thereafter,  $|\cdot|$  denotes the standard *Euclidean norm* in  $\mathbb{R}^3$  as well as the corresponding subordinated matrix norm in  $\mathbb{R}^{3 \times 3}$ . Notice that constants  $\bar{\sigma}_K$  and  $\underline{\sigma}_K$  measure the shape regularity of the hexahedron.

We recall that the *cofactor matrix* of  $\widehat{DF}_K(\hat{x})$  satisfies  $\text{Cof}[\widehat{DF}_K(\hat{x})]^t = J_K(\hat{x})[\widehat{DF}_K(\hat{x})]^{-1}$ . Explicit computations lead to the following bounds:

$$\underline{\sigma}_K^3 h_K^3 \leq J_K(\hat{x}) \leq \bar{\sigma}_K^3 h_K^3 \quad \text{and} \quad \underline{\sigma}_K^3 \bar{\sigma}_K^{-1} h_K^2 \leq |\text{Cof}(\widehat{DF}_K)(\hat{x})| \leq \bar{\sigma}_K^3 \underline{\sigma}_K^{-1} h_K^2 \quad \forall \hat{x} \in \widehat{K}. \quad (1)$$

To measure the ‘deviation’ of a hexahedron  $K$  with respect to a parallelepiped, we decompose  $F_K$  as a sum of two parts, one affine and the other depending on four geometrical vector parameters,  $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathbb{R}^3$  (see Fig. 1). Indeed, let  $\alpha_i := F_K(e_i) \in \mathbb{R}^3$ , with  $e_i$  being the  $i$ th canonical vector,  $i = 1, 2, 3$ . For  $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ , let  $F_K(\hat{x}) = F_K^l(\hat{x}) + F_K^d(\hat{x})$ , with

$$F_K^l(\hat{x}) = \hat{x}_1 \alpha_1 + \hat{x}_2 \alpha_2 + \hat{x}_3 \alpha_3 \quad \text{and} \quad F_K^d(\hat{x}) = \hat{x}_2 \hat{x}_3 \epsilon_1 + \hat{x}_1 \hat{x}_3 \epsilon_2 + \hat{x}_1 \hat{x}_2 \epsilon_3 + \hat{x}_1 \hat{x}_2 \hat{x}_3 (\epsilon_4 - \epsilon_1 - \epsilon_2 - \epsilon_3),$$

where  $F_K^l$  is the linear part and  $F_K^d$  the ‘deviation’ (for simplicity, we have not considered translations). Let us recall that if the faces of the element  $K$  are planar, then  $\epsilon_4 = \epsilon_1 + \epsilon_2 + \epsilon_3$  and the cubic term of  $F_K^d$  vanishes.

Let  $\varepsilon_K := \max_{1 \leq i \leq 4} |\epsilon_i|$ . The following lemma has been proved in [6]. Here and thereafter,  $C$  denotes a generic strictly positive constant not necessarily the same at each occurrence.

**Lemma 2.1.** *There exists a strictly positive constant  $C$  such that, for any hexahedron  $K$ ,*

$$\max_{\hat{x} \in \widehat{K}} |\widehat{\nabla} J_K(\hat{x})| \leq C \varepsilon_K h_K^2 \quad \text{and} \quad \max_{\hat{x} \in \widehat{K}} \left| \frac{\partial}{\partial \hat{x}_k} (\text{Cof}[\widehat{DF}_K(\hat{x})])_{ij} \right| \leq C \varepsilon_K h_K, \quad 1 \leq i, j, k \leq 3.$$

**Remark 1.** Similar estimates were proved by Thomas [15] in 2D. In this case there is a unique vector parameter  $\epsilon \in \mathbb{R}^2$  which measures the deviation of a quadrilateral  $K$  with respect to a parallelogram. Actually, Thomas used another one: the vector  $\delta$  joining the midpoints of the diagonals of  $K$ . However, it is simple to show that  $\delta = \frac{1}{2}\epsilon$ .

Given a vector field  $\mathbf{u}$  defined on  $K$ , Raviart–Thomas elements discretize the whole vector field instead of its components separately. The degrees of freedom of the lowest-order Raviart–Thomas hexahedron are the mean values of the fluxes of  $\mathbf{u}$  on each face of the hexahedron. To define these elements, we change coordinates to the reference unit cube and use the *Piola transform* associated with this change of coordinates (see for instance [11]):

$$\hat{\mathbf{u}}(\hat{x}) := J_K(\hat{x}) [\widehat{DF}_K(\hat{x})]^{-1} \mathbf{u}(F_K(\hat{x})), \quad \hat{x} \in \widehat{K}.$$

A lowest-order Raviart–Thomas element on  $K$  is a vector field  $\mathbf{u}$  whose Piola transform  $\hat{\mathbf{u}}$  satisfies

$$\hat{\mathbf{u}} \in \widehat{\mathcal{R}} := \mathcal{P}_{1,0,0}(\widehat{K}) \times \mathcal{P}_{0,1,0}(\widehat{K}) \times \mathcal{P}_{0,0,1}(\widehat{K}),$$

where  $\mathcal{P}_{i,j,k}(\widehat{K})$  denotes the set of polynomial functions defined on  $\widehat{K}$  of degrees  $i$ ,  $j$ , and  $k$  in  $\hat{x}_1$ ,  $\hat{x}_2$ , and  $\hat{x}_3$ , respectively. These vector fields have constant normal components on each face of the reference unit cube  $\widehat{K}$ . Since these constant normal components determine a unique vector field of this type, they can be taken as the degrees of freedom defining this element. Thus, the space of lowest-order Raviart–Thomas finite elements on  $K$  is

$$\mathcal{R}_K := \{ \mathbf{u} : K \rightarrow \mathbb{R}^3 : \hat{\mathbf{u}} \in \widehat{\mathcal{R}} \}.$$

### 3. Approximation properties

Given  $\mathbf{u} \in \mathbf{H}^r(K)^3$ ,  $r > \frac{1}{2}$ , the Raviart–Thomas interpolant of  $\mathbf{u}$  in  $K$  is defined as the unique  $\mathbf{u}_K \in \mathcal{R}_K$  such that  $\int_F \mathbf{u}_K \cdot \mathbf{n}_F = \int_F \mathbf{u} \cdot \mathbf{n}_F$  for all faces  $F$  of  $K$  ( $\mathbf{n}_F$  denotes a unit vector normal to  $F$ ). The mapping  $\mathbf{u} \mapsto \mathbf{u}_K$  extends by duality arguments to  $\mathbf{u} \in \mathbf{H}(\text{div}, K) \cap \mathbf{H}^r(K)^3 \forall r > 0$ , defining a bounded linear operator (see [11]).

Error estimates for this interpolation in 2D were proved in [15] for  $\mathbf{u} \in \mathbf{H}^1(K)^3$  with  $\text{div } \mathbf{u} \in \mathbf{H}^1(K)$ . The arguments in this reference have been adapted in [6] to 3D and fractional Sobolev spaces yielding the following result:

**Theorem 3.1.** *If  $\mathbf{u} \in \mathbf{H}(\text{div}, K) \cap \mathbf{H}^r(K)^3$ ,  $0 < r \leq 1$ , then  $\exists C > 0$ , only depending on  $\bar{\sigma}_K$ ,  $\underline{\sigma}_K$ , and  $r$ , such that*

$$\|\mathbf{u} - \mathbf{u}_K\|_{0,K} \leq C(\varepsilon_K h_K^{-1} \|\mathbf{u}\|_{0,K} + h_K^r |\mathbf{u}|_{r,K} + h_K \|\text{div } \mathbf{u}\|_{0,K}).$$

*If moreover  $\text{div } \mathbf{u} \in \mathbf{H}^s(K)$ ,  $0 < s \leq 1$ , then  $\exists C > 0$ , only depending on  $\bar{\sigma}_K$ ,  $\underline{\sigma}_K$ , and  $s$ , such that*

$$\|\text{div}(\mathbf{u} - \mathbf{u}_K)\|_{0,K} \leq C(\varepsilon_K h_K^{-1} \|\text{div } \mathbf{u}\|_{0,K} + h_K^s |\text{div } \mathbf{u}|_{s,K}).$$

**Remark 2.** If  $\mathbf{u} \in \mathbf{H}^r(K)^3$  with  $r > \frac{1}{2}$ , then the first estimate above can be improved as follows:

$$\|\mathbf{u} - \mathbf{u}_K\|_{0,K} \leq C(\varepsilon_K h_K^{-1} \|\mathbf{u}\|_{0,K} + h_K^r |\mathbf{u}|_{r,K}).$$

**Remark 3.** Girault and Raviart ([12], Theorem III.4.4) have extended some arguments from Thomas ([15], Theorem III.4.3) to prove the following improved estimate in 2D for  $\mathbf{u} \in \mathbf{H}^1(K)^3$ :

$$\|\mathbf{u} - \mathbf{u}_K\|_{0,K} \leq Ch_K |\mathbf{u}|_{1,K}.$$

More recently, Arnold et al. ([2], Theorem 7) have obtained a similar result valid for 2D Raviart–Thomas elements of arbitrary order. However, the arguments of these references do not seem to hold in 3D. Indeed, both rely on the fact that, in 2D, the columns of the matrix  $\text{Cof}(\widehat{DF}_K)^t$  belong to  $\mathcal{P}_{1,0} \times \mathcal{P}_{0,1}$ , which, in this case, corresponds to  $\widehat{\mathcal{R}}$  (see [2], Lemma 9). Instead, in 3D, these columns belong to  $\mathcal{P}_{2,1,1} \times \mathcal{P}_{1,2,1} \times \mathcal{P}_{1,1,2} \neq \widehat{\mathcal{R}}$ .

Next, consider a family of partitions in hexahedra  $\{\mathcal{T}_h\}$  of a given polyhedral bounded domain  $\Omega \subset \mathbb{R}^3$ ; as usual,  $h := \max_{K \in \mathcal{T}_h} h_K$  denotes the mesh-size. We recall that the family is *regular* if  $\exists \sigma_1, \sigma_2 > 0$ , the same for all meshes, such that  $\bar{\sigma}_K \leq \sigma_1$  and  $\underline{\sigma}_K \geq \sigma_2 \quad \forall K \in \mathcal{T}_h$ . On the other hand,  $\{\mathcal{T}_h\}$  is a family of *asymptotically parallelepiped* meshes, if for each element  $K \in \mathcal{T}_h$  there exists a parallelepiped such that the distances of the vertices of  $K$  to those of the parallelepiped are  $\mathcal{O}(h_K^2)$ . More precisely, if there exists a strictly positive constant  $\sigma_3$ , the same for all meshes, such that

$$\varepsilon_K \leq \sigma_3 h_K^2 \quad \forall K \in \mathcal{T}_h. \tag{2}$$

Notice that, as a consequence of Lemma 2.1, if  $\{\mathcal{T}_h\}$  is a family of asymptotically parallelepiped meshes, then the estimates for the derivatives of  $J_K$  and the entries of  $\text{Cof}(\widehat{DF}_K)$  are one order larger than those given by (1) for  $J_K$  and  $|\text{Cof}(\widehat{DF}_K)|$ , respectively.

Raviart–Thomas elements are  $\mathbf{H}(\text{div})$ -conforming in the following sense: Given  $\mathbf{u} \in \mathbf{H}(\text{div}, \Omega) \cap \mathbf{H}^r(\Omega)^3$  with  $r > 0$  (or  $\mathbf{u} \in \mathbf{H}^r(\Omega)^3$  with  $r > \frac{1}{2}$ ), let  $\mathbf{u}^R \in \mathbf{L}^2(\Omega)^3$  be defined by  $\mathbf{u}^R|_K := \mathbf{u}_K \quad \forall K \in \mathcal{T}_h$ ; then  $\mathbf{u}^R \in \mathbf{H}(\text{div}, \Omega)$ . Consequently,  $\mathbf{u}^R$ , which is called the Raviart–Thomas interpolant of  $\mathbf{u}$  in  $\Omega$ , belongs to

$$\mathcal{R}_h := \{\mathbf{u}_h \in \mathbf{H}(\text{div}, \Omega) : \mathbf{u}_h|_K \in \mathcal{R}_K \quad \forall K \in \mathcal{T}_h\}.$$

The following result is an immediate consequence of Theorem 3.1:

**Corollary 3.2.** *Let  $\mathbf{u} \in \mathbf{H}^r(\Omega)^3$  with  $0 < r \leq 1$  be such that  $\text{div } \mathbf{u} \in \mathbf{H}^s(\Omega)$  with  $0 < s \leq 1$ . Let  $\mathbf{u}^R \in \mathcal{R}_h$  be the Raviart–Thomas interpolant of  $\mathbf{u}$ . If  $\{\mathcal{T}_h\}$  is a regular family of asymptotically parallelepiped meshes of  $\Omega$ , then*

$$\|\mathbf{u} - \mathbf{u}^R\|_{0,\Omega} \leq Ch^r (\|\mathbf{u}\|_{r,\Omega} + \|\operatorname{div} \mathbf{u}\|_{0,\Omega}) \quad \text{and} \quad \|\operatorname{div}(\mathbf{u} - \mathbf{u}^R)\|_{0,\Omega} \leq Ch^s \|\operatorname{div} \mathbf{u}\|_{s,\Omega}.$$

**Remark 4.** The power 2 of  $h_K$  in the definition of asymptotically parallelepiped meshes (2) is somewhat arbitrary. Indeed, the corollary above holds true as far as  $\varepsilon_K \leq Ch_K^{1+\max\{r,s\}}$ .

#### 4. Need of the asymptotically parallelepiped assumption on the meshes

Hexahedral Raviart–Thomas elements have been successfully used to numerically solve the following spectral problem, which models the acoustic vibrations in a rigid cavity:

Find  $\lambda \in \mathbb{R}$  and  $\mathbf{u} \in H_0(\operatorname{div}, \Omega)$ ,  $\mathbf{u} \neq \mathbf{0}$ , such that

$$\int_{\Omega} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} = \lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \quad \forall \mathbf{v} \in H_0(\operatorname{div}, \Omega). \quad (3)$$

We recall that  $H_0(\operatorname{div}, \Omega) := \{\mathbf{v} \in H(\operatorname{div}, \Omega) : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$ , where  $\mathbf{n}$  is the outer unit vector normal to  $\partial\Omega$ .

The following discrete problem is obtained by using lowest-order Raviart–Thomas hexahedral elements:

Find  $\lambda_h \in \mathbb{R}$  and  $\mathbf{u}_h \in \mathcal{R}_h^0$ ,  $\mathbf{u}_h \neq \mathbf{0}$ , such that

$$\int_{\Omega} \operatorname{div} \mathbf{u}_h \operatorname{div} \mathbf{v}_h = \lambda_h \int_{\Omega} \mathbf{u}_h \cdot \mathbf{v}_h \quad \forall \mathbf{v}_h \in \mathcal{R}_h^0.$$

In the expression above,  $\mathcal{R}_h^0 := \mathcal{R}_h \cap H_0(\operatorname{div}, \Omega) = \{\mathbf{v}_h \in \mathcal{R}_h : \mathbf{v}_h \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$ .

For a family of meshes such that Corollary 3.2 holds true, the abstract spectral theory (see [4]) has been applied in [6] to prove convergence for eigenvalues and eigenfunctions. The asymptotically parallelepiped assumption is essential for this convergence to hold, as we show in the following example.

Consider a 2D cavity  $\Omega := [0, 1]^2$ . The eigenvalues of (3) are analytically known:

$$\lambda_{mn} = \pi^2(m^2 + n^2), \quad m, n = 0, 1, 2, \dots$$

To test the method with a family of non-asymptotically parallelogram meshes, we have used partitions of the domain into  $N \times N$  congruent trapezoids, all similar to the trapezoid with vertices  $(0, 0)$ ,  $(\frac{1}{2}, 0)$ ,  $(\frac{1}{2}, \frac{2}{3})$  and  $(0, \frac{1}{3})$ , as shown in Fig. 2. Clearly, the elements of these meshes are uniformly far from being parallelograms. They were used in [1–3] to show that the order of convergence of some quadrilateral finite elements deteriorates, even though shape-regular meshes are used. We have solved the problem with uniform meshes of  $N \times N$  squares, as well.

Tables 1 and 2 show the five smallest eigenvalues  $\lambda_{mn}$  computed with these square and trapezoid meshes, respectively. Each table includes the approximations of the eigenvalues obtained by extrapolating the computed ones, that we denote  $\hat{\lambda}_{mn}$  and  $\tilde{\lambda}_{mn}$ , respectively, as well as the computed orders of convergence. Table 1 also includes the exact eigenvalues and Table 2 the ratios between the extrapolated values obtained with each method.

We notice that the method converges in both cases with an almost perfect quadratic order. However, in the case of the trapezoid elements, it converges to wrong eigenvalues! This can only happen if  $\|\mathbf{u} - \mathbf{u}^R\|_{H(\operatorname{div}, \Omega)}$  does not converge to zero as  $h$  goes to zero. Thus, this numerical test shows that such convergence fails for these particular non-asymptotically parallelepiped meshes. A more detailed analysis of this phenomenon can be found in [6], including the reason why the quotients between exact and wrong eigenvalues are almost constant.

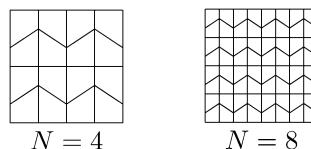


Fig. 2. Non-asymptotically parallelogram trapezoid meshes in  $\Omega = [0, 1]^2$ .

Table 1  
Eigenvalues computed with square Raviart–Thomas elements

	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$\hat{\lambda}_{mn}$	Order	Exact
$\lambda_{10}$	9.997081	9.901356	9.877535	9.871588	9.869585	2.00	9.869604
$\lambda_{01}$	9.997081	9.901356	9.877535	9.871588	9.869585	2.00	9.869604
$\lambda_{11}$	19.994162	19.802705	19.755066	19.743170	19.739168	2.00	19.739209
$\lambda_{20}$	41.546571	39.988324	39.605414	39.510139	39.479419	2.01	39.478418
$\lambda_{02}$	41.546571	39.988324	39.605414	39.510139	39.479419	2.01	39.478418

Table 2  
Eigenvalues computed with trapezoid Raviart–Thomas elements

	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$\hat{\lambda}_{mn}$	Order	$\tilde{\lambda}_{mn}/\hat{\lambda}_{mn}$
$\lambda_{10}$	10.347442	10.282911	10.266965	10.262896	10.261610	2.01	1.039721
$\lambda_{01}$	10.394167	10.294620	10.269891	10.263626	10.261597	2.00	1.039719
$\lambda_{11}$	20.736680	20.576279	20.536520	20.526438	20.523194	2.00	1.039719
$\lambda_{20}$	42.451339	41.390093	41.131825	41.067538	41.047158	2.02	1.039710
$\lambda_{02}$	43.196809	41.576587	41.178634	41.079252	41.046629	2.01	1.039697

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