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Mathematical Problems in Mechanics

Another approach to linear shell theory and a new proof of Korn's inequality on a surface

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Abstract

We propose a new approach to the quadratic minimization problems arising in Koiter's linear shell theory. The novelty consists in considering the linearized change of metric and change of curvature tensors as the new unknowns, instead of the displacement vector field as is customary. This approach also provides a new proof of Korn's inequality on a surface. ***To cite this article: P.G. Ciarlet, L. Gratie, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***
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Résumé

Une nouvelle approche de la théorie linéaire des coques et une nouvelle démonstration de l'inégalité de Korn sur une surface. On propose une nouvelle approche pour les problèmes quadratiques de minimisation rencontrés dans la théorie linéaire de coques de Koiter. La nouveauté consiste à considérer les tenseurs linéarisés de changement de métrique et de changement de courbure comme les nouvelles inconnues, au lieu du champ de déplacement comme à l'accoutumée. Cette approche conduit aussi à une nouvelle démonstration de l'inégalité de Korn sur une surface. ***Pour citer cet article : P.G. Ciarlet, L. Gratie, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***
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Version française abrégée

Les notations non définies ici le sont dans la version anglaise. Soit ω un domaine de $\mathbb{R}^2$. Selon Koiter [9], le problème de *traction pure* pour une *coque linéairement élastique* consiste à trouver un champ de vecteurs $\boldsymbol{\eta}^* = (\eta_i^*) \in \boldsymbol{V}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ tel que $j(\boldsymbol{\eta}^*) = \inf_{\boldsymbol{\eta} \in \boldsymbol{V}(\omega)} j(\boldsymbol{\eta})$, où

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$$
\begin{aligned}
&j(\boldsymbol{\eta}) := \frac{1}{2}\int_\omega \left\{\varepsilon \boldsymbol{A}\boldsymbol{\gamma}(\boldsymbol{\eta}) : \boldsymbol{\gamma}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3}\boldsymbol{A}\boldsymbol{\rho}(\boldsymbol{\eta}) : \boldsymbol{\rho}(\boldsymbol{\eta})\right\}\sqrt{a}\,\mathrm{d}y - l(\boldsymbol{\eta}),\\
&\boldsymbol{\gamma}(\boldsymbol{\eta}) := \big(\gamma_{\alpha\beta}(\boldsymbol{\eta})\big) \quad \text{où } \gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\beta \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta) \text{ et } \tilde{\boldsymbol{\eta}} := \eta_i \boldsymbol{a}^i,\\
&\boldsymbol{\rho}(\boldsymbol{\eta}) := \big(\rho_{\alpha\beta}(\boldsymbol{\eta})\big) \quad \text{où } \rho_{\alpha\beta}(\boldsymbol{\eta}) := (\partial_{\alpha\beta}\tilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma \tilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3,\\
&\boldsymbol{A}\boldsymbol{t} : \boldsymbol{t} := a^{\alpha\beta\sigma\tau} t_{\sigma\tau} t_{\alpha\beta} \quad \text{pour tout } \boldsymbol{t} = (t_{\alpha\beta}) \text{ et } l(\boldsymbol{\eta}) := \int_\omega p^i \eta_i \sqrt{a}\,\mathrm{d}\omega.
\end{aligned}
$$

Les trois *inconnues* η_i^* sont les composantes covariantes du champ de déplacement $\eta_i^* \boldsymbol{a}^i$ de la surface moyenne de la coque. Naturellement, on suppose que $l(\boldsymbol{\eta}) = 0$ pour tout $\boldsymbol{\eta} \in \mathbf{Rig}(\omega)$, où

$$
\mathbf{Rig}(\omega) := \big\{\boldsymbol{\eta} \in \boldsymbol{V}(\omega);\ \boldsymbol{\gamma}(\boldsymbol{\eta}) = \boldsymbol{\rho}(\boldsymbol{\eta}) = \mathbf{0} \text{ in } \boldsymbol{L}^2_{\mathrm{sym}}(\omega)\big\}.
$$

L'existence d'une telle solution « classique » repose sur l'*inégalité de Korn sur une surface*, comme il est rappelé dans les Théorèmes 1.1 et 1.2 ci-dessous.

L'objectif de cette Note est d'analyser une autre façon de poser le même problème, où les inconnues sont cette fois les six composantes covariantes $c^*_{\alpha\beta} := \gamma_{\alpha\beta}(\boldsymbol{\eta})$ et $r^*_{\alpha\beta} := \rho_{\alpha\beta}(\boldsymbol{\eta})$ des tenseurs linéarisés de changement de métrique et de changement de courbure. Selon la terminologie des mécaniciens (cf. Opoka et Pietraszkiewicz [10]), il s'agit donc d'identifier des *équations « intrinsèques »* en théorie des coques, dont le mérite est de donner directement les *efforts tranchants* et les *moments fléchissants*.

Le résultat principal de cette Note (Théorème 4.1) montre que cette approche conduit à résoudre le problème suivant de *minimisation avec contraintes* : Trouver

$$
(\boldsymbol{c}^*, \boldsymbol{r}^*) = \big(c_{\alpha\beta}, (r_{\alpha\beta})\big) \in \boldsymbol{T}(\omega) \text{ tel que } \kappa(\boldsymbol{c}^*, \boldsymbol{r}^*) = \inf\big\{\kappa(\boldsymbol{c}, \boldsymbol{r}); (\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{T}(\omega)\big\},
$$

où

$$
\begin{aligned}
&\kappa(\boldsymbol{c}, \boldsymbol{r}) := \frac{1}{2}\int_\omega \left\{\varepsilon \boldsymbol{A}\boldsymbol{c} : \boldsymbol{c} + \frac{\varepsilon^3}{3}\boldsymbol{A}\boldsymbol{r} : \boldsymbol{r}\right\}\sqrt{a}\,\mathrm{d}y - l^\flat(\boldsymbol{c}, \boldsymbol{r}),\\
&\boldsymbol{T}(\omega) := \big\{(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega);\ \boldsymbol{R}(\boldsymbol{c}, \boldsymbol{r}) = \mathbf{0} \text{ in } \boldsymbol{H}^{-2}(\widehat{\Omega})\big\},
\end{aligned}
$$

où $\boldsymbol{R} \in \mathcal{L}(\boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega); \boldsymbol{H}^{-2}(\widehat{\Omega}))$, $\widehat{\Omega} = \omega \times \left]-\varepsilon_0, \varepsilon_0\right[$ pour $\varepsilon_0 > 0$ convenable, et $l^\flat := l \circ \boldsymbol{H}$ où $\boldsymbol{H}$ est un isomorphisme de l'espace $\boldsymbol{T}(\omega)$ sur l'espace quotient $\boldsymbol{V}(\omega)/\mathbf{Rig}(\omega)$.

La difficulte principale (cf. Théorème 2.3) consiste à établir qu'il existe effectivement une telle application $\boldsymbol{R}$, ayant par conséquent la propriété suivante : Un couple $(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega)$ vérifie $\boldsymbol{R}(\boldsymbol{c}, \boldsymbol{r}) = \mathbf{0}$ dans $\boldsymbol{H}^{-2}(\widehat{\Omega})$ si, et seulement si, il existe $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ tel que $\boldsymbol{c} = \boldsymbol{\gamma}(\boldsymbol{\eta})$ et $\boldsymbol{r} = \boldsymbol{\rho}(\boldsymbol{\eta})$ dans $\boldsymbol{L}^2_{\mathrm{sym}}(\omega)$. L'un des outils essentiels de la démonstration est un résultat de Ciarlet et Ciarlet, Jr. [5] (cf. Théorème 2.1), établi pour traiter d'autres équations « intrinsèques », en élasticité linéarisée tri-dimensionnelle cette fois.

Il est à noter que cette approche conduit à une *démonstration nouvelle de l'inégalité de Korn sur une surface*, reposant sur une version « faible » d'un *théorème classique de Poincaré* (cf. Théorème 2.2), alors que sa démonstration « classique » repose sur un *lemme fondamental de J.L. Lions* (cf. Théorèmes 1.1 et 1.2).

Les démonstrations détaillées sont données dans [6].

1. The classical approach to linear shell theory

Greek indices, and Latin indices range in the set $\{1, 2\}$ and $\{1, 2, 3\}$, respectively. The summation convention with respect to repeated indices is used in conjunction with these rules.

The notation $\mathbb{E}^3$ designates a *three-dimensional Euclidean space*, with vectors $\hat{\boldsymbol{e}}^i$ forming an orthonormal basis. The Euclidean norm of $\boldsymbol{a} \in \mathbb{E}^3$ is denoted $|\boldsymbol{a}|$ and the Euclidean and exterior products of $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^3$ are denoted $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$.

A generic point in $\mathbb{R}^2$ will be denoted $y = (y_\alpha)$; then $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$. A generic point in $\mathbb{R}^3$ will be denoted $x = (x_i)$; then $\partial_i := \partial/\partial x_i$ and $\partial_{ij} := \partial^2/\partial x_i \partial x_j$. Given a smooth enough vector field $\boldsymbol{v} = (v_i)$ defined on a subset of $\mathbb{R}^3$, the 3×3 matrix field with $\partial_j v_i$ as its element at the i-th row and j-th column is denoted $\boldsymbol{\nabla v}$. A generic point in $\mathbb{E}^3$ will be denoted $\hat{x} = (\hat{x}_i)$; then $\hat{\partial}_i := \partial/\partial \hat{x}_i$ and $\hat{\partial}_{ij} := \partial^2/\partial \hat{x}_i \partial \hat{x}_j$; the notation $\widehat{\boldsymbol{\nabla}} \hat{\boldsymbol{v}}$ should be self-explanatory. The coordinates $\hat{x}_i$ of a point $\hat{x} \in \mathbb{E}^3$ will be referred to as *Cartesian coordinates*.

A *domain* U in $\mathbb{R}^n$, $n \geqslant 2$, or in $\mathbb{E}^3$, is an open, bounded, connected subset with a Lipschitz-continuous boundary, the set U being locally on the same side of its boundary. Spaces of vector-valued, or matrix-valued, functions over U are denoted by boldface letters, and the norms of the spaces $L^2(U)$ or $\boldsymbol{L}^2(U)$, and $H^m(U)$ or $\boldsymbol{H}^m(U)$, $m \geqslant 1$, are denoted $\|\cdot\|_{0,U}$, and $\|\cdot\|_{m,U}$.

Given a domain ω in $\mathbb{R}^2$ and an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$, define the *surface* $S := \boldsymbol{\theta}(\overline{\omega})$. The covariant components $a_{\alpha\beta} = a_{\beta\alpha} \in \mathcal{C}^2(\overline{\omega})$ and $b_{\alpha\beta} = b_{\beta\alpha} \in \mathcal{C}^1(\overline{\omega})$ of the *first*, and *second, fundamental forms* of the surface S are then respectively given by $a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ and $b_{\alpha\beta} := \boldsymbol{a}_3 \cdot \partial_\alpha \boldsymbol{a}_\beta$, where $\boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta}$ and $\boldsymbol{a}_3 := (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2)/|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|$. We also let $a := \det(a_{\alpha\beta}) \in \mathcal{C}^2(\overline{\omega})$.

Two other fundamental tensors play a key rôle in the two-dimensional theory of linearly elastic shells, the *linearized change of metric tensor* and the *linearized change of curvature tensor*, each one being associated with a displacement vector field $\tilde{\boldsymbol{\eta}} := \eta_i \boldsymbol{a}^i$ of the surface S, where

$$\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega),$$

and the vector fields $\boldsymbol{a}^i$ are defined by the relations $\boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta^i_j$. The covariant components of these tensors are given by

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\beta \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta) \quad \text{and} \quad \rho_{\alpha\beta}(\boldsymbol{\eta}) := (\partial_{\alpha\beta} \tilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \tilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3,$$

where $\Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\alpha \boldsymbol{a}_\beta$. Note that $\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ and $\rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ if $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$.

Consider a *linearly elastic shell* with *middle surface* S and *thickness* $2\varepsilon > 0$, whose *two-dimensional elasticity tensor* is *uniformly positive definite*, in the sense that there exists a constant $b > 0$ such that its contravariant components $a^{\alpha\beta\sigma\tau}$ satisfy $b \sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \leqslant a^{\alpha\beta\sigma\tau}(y) t_{\sigma\tau} t_{\alpha\beta}$ for all $y \in \overline{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$ of order two. Assume that the shell is subjected to *applied forces* whose resultant after integration across the thickness of the shell has contravariant components $p^i \in L^2(\omega)$. Assume, finally, that the lateral face of the shell is free. In other words, we are considering a *pure traction problem for a linearly elastic shell*.

As a mathematical model for this problem, we select the well-known *two-dimensional Koiter equations* (so named after Koiter [9]), in the form of the following *quadratic minimization problem*: The unknowns are the three covariant components $\eta^*_i : \overline{\omega} \to \mathbb{R}$ of the displacement field $\eta^*_i \boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ of the middle surface S of the shell and the vector field $\boldsymbol{\eta}^* := (\eta^*_i)$ satisfies $\boldsymbol{\eta}^* \in \boldsymbol{V}(\omega)$ and $j(\boldsymbol{\eta}^*) = \inf_{\boldsymbol{\eta} \in \boldsymbol{V}(\omega)} j(\boldsymbol{\eta})$, where

$$j(\boldsymbol{\eta}) := \frac{1}{2} \int_\omega \left\{ \varepsilon \boldsymbol{A} \boldsymbol{\gamma}(\boldsymbol{\eta}) : \boldsymbol{\gamma}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} \boldsymbol{A} \boldsymbol{\rho}(\boldsymbol{\eta}) : \boldsymbol{\rho}(\boldsymbol{\eta}) \right\} \sqrt{a}\, \mathrm{d}y - l(\boldsymbol{\eta}),$$

$$\boldsymbol{\gamma}(\boldsymbol{\eta}) := \big(\gamma_{\alpha\beta}(\boldsymbol{\eta})\big) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega) := \left\{ \boldsymbol{c} = (c_{\alpha\beta}) \in \big(L^2(\omega)\big)^4;\ c_{\alpha\beta} = c_{\beta\alpha} \right\} \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}(\omega),$$

$$\boldsymbol{\rho}(\boldsymbol{\eta}) := \big(\rho_{\alpha\beta}(\boldsymbol{\eta})\big) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega) \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}(\omega),$$

$$\boldsymbol{A}\boldsymbol{t} : \boldsymbol{t} := a^{\alpha\beta\sigma\tau} t_{\sigma\tau} t_{\alpha\beta} \in L^1(\omega) \quad \text{for all } \boldsymbol{t} = (t_{\alpha\beta}) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega),$$

$$l(\boldsymbol{\eta}) := \int_\omega p^i \eta_i \sqrt{a}\, \mathrm{d}\omega \quad \text{for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega).$$

Assume that the linear form l satisfies the (clearly necessary) *compatibility conditions* $l(\boldsymbol{\eta}) = 0$ for all $\boldsymbol{\eta} \in \mathbf{Rig}(\omega)$, where

$$\mathbf{Rig}(\omega) := \left\{\boldsymbol{\eta} \in \boldsymbol{V}(\omega);\, \boldsymbol{\gamma}(\boldsymbol{\eta}) = \boldsymbol{\rho}(\boldsymbol{\eta}) = \mathbf{0} \text{ in } \boldsymbol{L}^2_{\mathrm{sym}}(\omega)\right\}.$$

The above minimization problem then amounts to finding an equivalence class $\dot{\boldsymbol{\eta}}^*$ that satisfies

$$\dot{\boldsymbol{\eta}}^* \in \dot{\boldsymbol{V}}(\omega) := \boldsymbol{V}(\omega)/\mathbf{Rig}(\omega) \quad \text{and} \quad j(\dot{\boldsymbol{\eta}}^*) = \inf_{\dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega)} j(\dot{\boldsymbol{\eta}}).$$

Let

$$\left\|(\boldsymbol{c}, \boldsymbol{r})\right\|_{0,\omega} := \left\{\sum_{\alpha,\beta} \|c_{\alpha\beta}\|^2_{0,\omega} + \sum_{\alpha,\beta} \|r_{\alpha\beta}\|^2_{0,\omega}\right\}^{1/2}$$

for all $(\boldsymbol{c}, \boldsymbol{r}) = ((c_{\alpha\beta}), (r_{\alpha\beta})) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega)$,

$$\|\boldsymbol{\eta}\|_{\boldsymbol{V}(\omega)} := \left\{\sum_{\alpha} \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega}\right\}^{1/2} \quad \text{for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega),$$

$$\|\dot{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{V}}(\omega)} := \inf_{\boldsymbol{\xi} \in \mathbf{Rig}(\omega)} \|\boldsymbol{\eta} + \boldsymbol{\xi}\|_{\boldsymbol{V}(\omega)} \quad \text{for all } \dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega).$$

In order to establish the existence and uniqueness of a minimizer of the functional j over the space $\dot{\boldsymbol{V}}(\omega)$, it suffices, thanks to the positive-definiteness of the two-dimensional elasticity tensor of the shell, to show that *the mapping* $\dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega) \to \|(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}}))\|_{0,\omega}$ *is a norm over the quotient space* $\dot{\boldsymbol{V}}(\omega)$, *equivalent to the quotient norm* $\|\cdot\|_{\dot{\boldsymbol{V}}(\omega)}$. To prove that this is indeed the case is achieved in two stages. The first stage, which is due to Bernadou and Ciarlet [2] (see also Bernadou, Ciarlet and Miara [3]), consists in establishing a first basic *Korn inequality on a surface*, "*over the space* $\boldsymbol{V}(\omega)$":

Theorem 1.1. *Let there be given a domain ω in $\mathbb{R}^2$ and an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$. Then there exists a constant $c = c(\omega, \boldsymbol{\theta})$ such that*

$$\|\boldsymbol{\eta}\|_{\boldsymbol{V}(\omega)} \leqslant c\left\{\sum_{\alpha} \|\eta_\alpha\|^2_{0,\omega} + \|\eta_3\|^2_{1,\omega} + \left\|\big(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{\eta})\big)\right\|^2_{0,\omega}\right\}^{1/2} \quad \textit{for all } \boldsymbol{\eta} \in \boldsymbol{V}(\omega).$$

As shown in *ibid.*, this inequality essentially relies on a fundamental *lemma of J.L. Lions*: Let U be a domain in $\mathbb{R}^n$. If a distribution $v \in H^{-1}(U)$ has its n first partial derivatives also in $H^{-1}(U)$, then $v \in L^2(\Omega)$ (see Theorem 3.2, Chapter 3 of Duvaut and Lions [8] for domains with smooth boundaries and Amrouche and Girault [1] for domains with Lipschitz-continuous boundaries).

The second stage consists in establishing another basic *Korn's inequality on a surface, this time* "*over the quotient space* $\dot{\boldsymbol{V}}(\omega)$" (the proof, by contradiction, uses the finite dimensionality of the space $\mathbf{Rig}(\omega)$, Rellich theorem, and the Korn inequality of Theorem 1.1):

Theorem 1.2. *Let there be given a domain ω in $\mathbb{R}^2$ and an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$. Then there exists a constant $\dot{c} = \dot{c}(\omega, \boldsymbol{\theta})$ such that*

$$\|\dot{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{V}}(\omega)} \leqslant \dot{c}\left\|\big(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}})\big)\right\|_{0,\omega} \quad \textit{for all } \dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega).$$

Our subsequent analysis will provide 'as by-products' entirely different proofs of the above Korn inequalities on a surface; see Theorem 3.2.

2. A necessary and sufficient condition for matrix fields to be linearized change of metric and change of curvature tensors

First, we recall the methodology of Ciarlet and Ciarlet, Jr. [5] for characterizing those symmetric 3×3 matrix fields $\hat{\boldsymbol{e}}$ with components in $L^2(\widehat{\Omega})$, where $\widehat{\Omega}$ is a domain in $\mathbb{R}^3$, that can be written for some $\hat{\boldsymbol{v}} \in \boldsymbol{H}^1(\widehat{\Omega})$ as $\hat{\boldsymbol{e}} = \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}})$, where $\hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}}) := \frac{1}{2}(\widehat{\boldsymbol{\nabla}}\hat{\boldsymbol{v}}^{\mathrm{T}} + \widehat{\boldsymbol{\nabla}}\hat{\boldsymbol{v}})$ denotes the *linearized strain tensor in Cartesian coordinates* associated with the displacement field $\hat{v}_i \hat{\boldsymbol{e}}^i$ of the set $\widehat{\Omega}$. More specifically, it is shown in *ibid*. that the classical *St Venant compatibility relations* are also sufficient conditions *in the sense of distributions*, according to the following result:

Theorem 2.1. *Let $\widehat{\Omega}$ be a simply-connected domain in $\mathbb{R}^3$. Let $\hat{\boldsymbol{e}} = (\hat{e}_{ij})$ be a symmetric matrix field with components $\hat{e}_{ij} \in L^2(\widehat{\Omega})$ that satisfy the following compatibility relations*:

$$\widehat{\mathcal{R}}_{ijkl}(\hat{\boldsymbol{e}}) := \hat{\partial}_{lj}\hat{e}_{ik} + \hat{\partial}_{ki}\hat{e}_{jl} - \hat{\partial}_{li}\hat{e}_{jk} - \hat{\partial}_{kj}\hat{e}_{il} = 0 \quad \text{in } H^{-2}(\widehat{\Omega})$$

(*it is easily verified that the eighty-one relations $\widehat{\mathcal{R}}_{ijkl}(\hat{\boldsymbol{e}}) = 0$ in $H^{-2}(\widehat{\Omega})$ are satisfied if only six of them hold, provided they are suitably chosen*). *Then there exists a vector field $\hat{\boldsymbol{v}} = (\hat{v}_i) \in \boldsymbol{H}^1(\widehat{\Omega})$ such that $\hat{\boldsymbol{e}} = \hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{v}})$ in $\boldsymbol{L}^2(\widehat{\Omega})$.*

The proof relies on the following *weak version of a classical theorem of Poincaré* (see *ibid*.):

Theorem 2.2. *Let $\widehat{\Omega}$ be a simply-connected domain in $\mathbb{R}^3$. Let $\hat{h}_k \in H^{-1}(\widehat{\Omega})$ be distributions that satisfy $\hat{\partial}_l \hat{h}_k = \hat{\partial}_k \hat{h}_l$ in $H^{-2}(\widehat{\Omega})$. Then there exists a function $\hat{p} \in L^2(\widehat{\Omega})$, unique up to an additive constant, such that $\hat{h}_k = \hat{\partial}_k \hat{p}$ in $H^{-1}(\widehat{\Omega})$.*

Thanks to Theorem 2.1, we can establish an analog characterization, but this time in the case of a surface.

Theorem 2.3. *Let there be given a simply-connected domain ω in $\mathbb{R}^2$ and an injective immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$. For each $\varepsilon > 0$, define the mapping $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{\omega} \times [-\varepsilon, \varepsilon]; \mathbb{E}^3)$ by*

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \frac{\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y)}{|\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y)|} \quad \text{for all } (y, x_3) \in \overline{\omega} \times [-\varepsilon, \varepsilon].$$

Then there exist $\varepsilon_0 > 0$ and a mapping $\boldsymbol{R} \in \mathcal{L}(\boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega); \boldsymbol{H}^{-2}(\widehat{\Omega}))$, where $\widehat{\Omega} := \boldsymbol{\Theta}(\omega \times]-\varepsilon_0, \varepsilon_0[)$ and $\boldsymbol{H}^{-2}(\widehat{\Omega}) := (H^{-2}(\widehat{\Omega}))^6$, with the following property: A pair $(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega)$ of symmetric matrix fields satisfies $\boldsymbol{R}(\boldsymbol{c}, \boldsymbol{r}) = \boldsymbol{0}$ in $\boldsymbol{H}^{-2}(\widehat{\Omega})$ if and only if there exists a vector field $\boldsymbol{\eta} \in \boldsymbol{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ such that

$$\boldsymbol{c} = \boldsymbol{\gamma}(\boldsymbol{\eta}) \quad \text{and} \quad \boldsymbol{r} = \boldsymbol{\rho}(\boldsymbol{\eta}) \quad \text{in } \boldsymbol{L}^2_{\mathrm{sym}}(\omega).$$

In this case, all other solutions $\boldsymbol{\eta}' \in \boldsymbol{V}(\omega)$ of the equations $\boldsymbol{c} = \boldsymbol{\gamma}(\boldsymbol{\eta}')$ and $\boldsymbol{r} = \boldsymbol{\rho}(\boldsymbol{\eta}')$ in $\boldsymbol{L}^2_{\mathrm{sym}}(\omega)$ are such that $(\boldsymbol{\eta}' - \boldsymbol{\eta}) \in \mathbf{Rig}(\omega)$.

Sketch of proof. The uniqueness up to vector fields in the space $\mathbf{Rig}(\omega)$ is well known. Otherwise the outline of the proof is as follows (see Ciarlet and Gratie [6] for a detailed proof):

(i) Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an injective immersion. Let $\boldsymbol{a}_i$ and $\boldsymbol{a}^j$ denote the vector fields defined by $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$, $\boldsymbol{a}_3 = (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2)/|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|$, and $\boldsymbol{a}^j \cdot \boldsymbol{a}_i = \delta_i^j$. Then (cf. Ciarlet [4, Theorem 3.1-1]) there exists $\varepsilon_0 > 0$ such that the mapping $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbb{E}^3$ defined by $\boldsymbol{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y)$ for all $(y, x_3) \in \overline{\Omega}$, where $\Omega := \omega \times]-\varepsilon_0, \varepsilon_0[$, is a $\mathcal{C}^2$-diffeomorphism from $\overline{\Omega}$ onto its image $\boldsymbol{\Theta}(\overline{\Omega})$. Let $\boldsymbol{g}_i$ and $\boldsymbol{g}^j$ denote the vector fields defined by the relations $\boldsymbol{g}_i = \partial_i \boldsymbol{\Theta}$ and $\boldsymbol{g}^j \cdot \boldsymbol{g}_i = \delta_i^j$ and let the symmetric matrix field $\boldsymbol{\varepsilon}(\boldsymbol{v}) := (\varepsilon_{ij}(\boldsymbol{v})) \in \boldsymbol{L}^2_{\mathrm{sym}}(\Omega)$, where $\varepsilon_{ij}(\boldsymbol{v}) := \frac{1}{2}(\partial_j v_i + \partial_i v_j) - \Gamma_{ij}^p v_p$ and $\Gamma_{ij}^p := \boldsymbol{g}^p \cdot \partial_i \boldsymbol{g}_j$, denotes the *linearized strain tensor field in curvilinear coordinates* associated with any displacement field $v_i \boldsymbol{g}^i$ of the set $\boldsymbol{\Theta}(\overline{\Omega})$ such that $v_i \in H^1(\Omega)$.

With any vector field $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega)$, let there be associated the vector field $\boldsymbol{v} = (v_j) \in \boldsymbol{H}^1(\Omega)$ defined by

$$v_j(y, x_3)\boldsymbol{g}^j(y, x_3) = \eta_i(y)\boldsymbol{a}^i(y) - x_3\big(\partial_\alpha \eta_3(y) + b^\sigma_\alpha(y)\eta_\sigma(y)\big)\boldsymbol{a}^\alpha(y)$$

for all $(y, x_3) \in \Omega$, where $b^\sigma_\alpha := a^{\beta\sigma} b_{\alpha\beta}$. Then (cf. Ciarlet and S. Mardare [7]) *the linear mapping $\boldsymbol{F} : \boldsymbol{V}(\omega) \to \boldsymbol{H}^1(\Omega)$ defined by $\boldsymbol{F}(\boldsymbol{\eta}) := \boldsymbol{v}$ is an isomorphism from the space $\boldsymbol{V}(\omega)$ onto the Hilbert space $\boldsymbol{V}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \varepsilon_{i3}(\boldsymbol{v}) = 0$ in $\Omega\}$*. Besides,

$$\varepsilon_{\alpha\beta}\big(\boldsymbol{F}(\boldsymbol{\eta})\big) = \gamma_{\alpha\beta}(\boldsymbol{\eta}) - x_3\rho_{\alpha\beta}(\boldsymbol{\eta}) + \frac{x_3^2}{2}\big\{b^\sigma_\alpha \rho_{\beta\sigma}(\boldsymbol{\eta}) + b^\tau_\beta \rho_{\alpha\tau}(\boldsymbol{\eta}) - 2b^\sigma_\alpha b^\tau_\beta \gamma_{\sigma\tau}(\boldsymbol{\eta})\big\}.$$

(ii) Given any vector field $\hat{\boldsymbol{v}} = (\hat{v}_i) \in \boldsymbol{H}^1(\widehat{\Omega})$, let the vector field $\boldsymbol{v} = (v_j) \in \boldsymbol{H}^1(\Omega)$ be defined by means of the relations $v_j(x)\boldsymbol{g}^j(x) = \hat{v}_i(\hat{x})\hat{\boldsymbol{e}}^i$ for almost all $x = \boldsymbol{\Theta}^{-1}(\hat{x}) \in \Omega$. Let $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega)$, let the mapping $\mathcal{B} \in \mathcal{L}(\boldsymbol{L}^2_{\mathrm{sym}}(\Omega); \boldsymbol{L}^2_{\mathrm{sym}}(\widehat{\Omega}))$ be defined for any $\boldsymbol{e} \in \boldsymbol{L}^2_{\mathrm{sym}}(\Omega)$ by

$$(\mathcal{B}\boldsymbol{e})(\hat{x}) := (\boldsymbol{\nabla}\boldsymbol{\Theta}^{-\mathrm{T}}\boldsymbol{e}\boldsymbol{\nabla}\boldsymbol{\Theta}^{-1})(x) \quad \text{for almost all } \hat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega},$$

and let, for any $\hat{\boldsymbol{e}} = (\hat{e}_{ij}) \in \boldsymbol{L}^2_{\mathrm{sym}}(\widehat{\Omega})$,

$$\widehat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{e}}) := \big(\widehat{\mathcal{R}}_{ijkl}(\hat{\boldsymbol{e}})\big), \quad \text{where } \widehat{\mathcal{R}}_{ijkl}(\hat{\boldsymbol{e}}) := \hat{\partial}_{lj}\hat{e}_{ik} + \hat{\partial}_{ki}\hat{e}_{jl} - \hat{\partial}_{li}\hat{e}_{jk} - \hat{\partial}_{kj}\hat{e}_{il}.$$

Then *a matrix field $\boldsymbol{e} \in \boldsymbol{L}^2_{\mathrm{sym}}(\Omega)$ is such that there exists a vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$ that satisfies $\boldsymbol{e} = \boldsymbol{\varepsilon}(\boldsymbol{v})$ in $\boldsymbol{L}^2_{\mathrm{sym}}(\Omega)$ if and only if $\boldsymbol{\mathcal{R}}(\boldsymbol{e}) = \boldsymbol{0}$ in $\boldsymbol{H}^{-2}(\widehat{\Omega}) = (H^{-2}(\widehat{\Omega}))^6$, where the mapping $\boldsymbol{\mathcal{R}} \in \mathcal{L}(\boldsymbol{L}^2_{\mathrm{sym}}(\Omega); \boldsymbol{H}^{-2}(\widehat{\Omega}))$ is defined by $\boldsymbol{\mathcal{R}} := \widehat{\boldsymbol{\mathcal{R}}} \circ \boldsymbol{\mathcal{B}}$.*

(iii) Let the mapping $\boldsymbol{G} \in \mathcal{L}(\boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega); \boldsymbol{L}^2_{\mathrm{sym}}(\Omega))$ be defined by

$$\big(\boldsymbol{G}(\boldsymbol{c}, \boldsymbol{r})\big)_{\alpha\beta} = c_{\alpha\beta} - x_3 r_{\alpha\beta} + \frac{x_3^2}{2}\{b^\sigma_\alpha r_{\beta\sigma} + b^\tau_\beta r_{\alpha\tau} - 2b^\sigma_\alpha b^\tau_\beta c_{\sigma\tau}\},$$
$$\big(\boldsymbol{G}(\boldsymbol{c}, \boldsymbol{r})\big)_{i3} = 0,$$

for any $(\boldsymbol{c}, \boldsymbol{r}) = ((c_{\alpha\beta}), (r_{\alpha\beta})) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega)$. Then *a pair $(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega)$ of matrix fields is such that there exists a vector field $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ that satisfies $\boldsymbol{c} = \boldsymbol{\gamma}(\boldsymbol{\eta})$ and $\boldsymbol{r} = \boldsymbol{\rho}(\boldsymbol{\eta})$ in $\boldsymbol{L}^2_{\mathrm{sym}}(\omega)$ if and only if $\boldsymbol{R}(\boldsymbol{c}, \boldsymbol{r}) = \boldsymbol{0}$ in $\boldsymbol{H}^{-2}(\widehat{\Omega}) = (H^{-2}(\widehat{\Omega}))^6$, where the mapping $\boldsymbol{R} \in \mathcal{L}(\boldsymbol{L}^2_{\mathrm{sym}}(\Omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\Omega); \boldsymbol{H}^{-2}(\widehat{\Omega}))$ is defined by $\boldsymbol{R} = \boldsymbol{\mathcal{R}} \circ \boldsymbol{G}$, the mapping $\boldsymbol{R} \in \mathcal{L}(\boldsymbol{L}^2_{\mathrm{sym}}(\Omega); \boldsymbol{H}^{-2}(\widehat{\Omega}))$ being that defined in* (ii). □

3. A new proof of Korn's inequalities on a surface

Complete proofs of Theorems 3.1 and 3.2 are found in Ciarlet and Gratie [6].

Thanks to Theorem 2.3, we first define in a natural way a basic isomorphism, which plays a key rôle in the rest of this Note.

Theorem 3.1. *Let there be given a simply-connected domain ω in $\mathbb{R}^2$ and an injective immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$. Define the space*

$$\boldsymbol{T}(\omega) := \big\{(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega);\ \boldsymbol{R}(\boldsymbol{c}, \boldsymbol{r}) = \boldsymbol{0} \text{ in } \boldsymbol{H}^{-2}(\widehat{\Omega})\big\},$$

where the open set $\widehat{\Omega}$ and the mapping $\boldsymbol{R} \in \mathcal{L}(\boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega); \boldsymbol{H}^{-2}(\widehat{\Omega}))$ are defined as in Theorem 2.3*. Given any element $(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{T}(\omega)$, there exists, again by Theorem* 2.3*, a unique equivalence class $\dot{\boldsymbol{\eta}}$ in the quotient space $\dot{\boldsymbol{V}}(\omega)$ that satisfies $\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}) = \boldsymbol{c}$ and $\boldsymbol{\rho}(\dot{\boldsymbol{\eta}}) = \boldsymbol{r}$ in $\boldsymbol{L}^2_{\mathrm{sym}}(\omega)$. Then the mapping $\boldsymbol{H} : \boldsymbol{T}(\omega) \to \dot{\boldsymbol{V}}(\omega)$ defined by $\boldsymbol{H}(\boldsymbol{c}, \boldsymbol{r}) := \dot{\boldsymbol{\eta}}$ is an isomorphism between the Hilbert spaces $\boldsymbol{T}(\omega)$ and $\dot{\boldsymbol{V}}(\omega)$.*

Sketch of proof. It is easily seen that the mapping $\boldsymbol{H}$ is injective and surjective and that its inverse mapping is continuous. The conclusion then follows from the *closed graph theorem*. $\square$

Remarkably, the two Korn inequalities on a surface recalled earlier in Theorems 1.1 and 1.2, can now be recovered as simple corollaries to Theorem 3.1.

Theorem 3.2. *Let there be given a simply-connected domain ω in $\mathbb{R}^2$ and an injective immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$. That the mapping $\boldsymbol{H} : \boldsymbol{T}(\omega) \to \dot{\boldsymbol{V}}(\omega)$ is an isomorphism implies both Korn's inequalities on a surface, i.e., "over the space $\boldsymbol{V}(\omega)$" (Theorem 1.1) and "over the quotient space $\dot{\boldsymbol{V}}(\omega)$" (Theorem 1.2).*

Sketch of proof. (i) Since $\boldsymbol{H}$ is an isomorphism, there exists a constant $\dot{c}$ such that $\|\boldsymbol{H}(\boldsymbol{c}, \boldsymbol{r})\|_{\dot{\boldsymbol{V}}(\omega)} \leqslant \dot{c}\|(\boldsymbol{c}, \boldsymbol{r})\|_{0,\omega}$ for all $(\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{T}(\omega)$, or equivalently, again because $\boldsymbol{H}$ is an isomorphism, such that $\|\dot{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{V}}(\omega)} \leqslant \dot{c}\|(\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}}))\|_{0,\omega}$ for all $\dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega)$. This inequality is exactly *Korn's inequality* "*over the quotient space* $\dot{\boldsymbol{V}}(\omega)$" of Theorem 1.2.

(ii) One can then show, by means of an argument by contradiction, which uses in an essential manner the finite dimensionality of the space $\mathbf{Rig}(\omega)$, that this Korn inequality in turn implies *Korn's inequality* "*over the space* $\boldsymbol{V}(\omega)$" of Theorem 1.1. $\square$

4. A new approach to existence theory for Koiter's linear shell equations

Thanks again to the isomorphism $\boldsymbol{H}$ introduced in Theorem 3.1, the quadratic minimization problem that models the pure traction problem of a linearly elastic shell (see Section 1) can be recast as *another quadratic minimization problem*, this time over the space $\boldsymbol{T}(\omega)$, also introduced in Theorem 3.1. The proof of the next theorem is straightforward.

Theorem 4.1. *Given a simply-connected domain ω in $\mathbb{R}^2$ and an injective immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$, define the Hilbert space $\boldsymbol{T}(\omega)$ as in Theorem 3.1, and define the quadratic functional $\kappa : \boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega) \to \mathbb{R}$ by*

$$\kappa(\boldsymbol{c}, \boldsymbol{r}) := \frac{1}{2}\int_\omega \left\{\varepsilon \boldsymbol{A}\boldsymbol{c} : \boldsymbol{c} + \frac{\varepsilon^3}{3}\boldsymbol{A}\boldsymbol{r} : \boldsymbol{r}\right\}\sqrt{a}\,\mathrm{d}y - l^\flat(\boldsymbol{c}, \boldsymbol{r})$$

for all $(\boldsymbol{c}, \boldsymbol{r}) = ((c_{\alpha\beta}), (r_{\alpha\beta})) \in \boldsymbol{L}^2_{\mathrm{sym}} \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega)$, where $l^\flat := l \circ \boldsymbol{H}$. Then the minimization problem: Find $(\boldsymbol{c}^*, \boldsymbol{r}^*) \in \boldsymbol{T}(\omega)$ such that

$$\kappa(\boldsymbol{c}^*, \boldsymbol{r}^*) = \inf\bigl\{\kappa(\boldsymbol{c}, \boldsymbol{r});\ (\boldsymbol{c}, \boldsymbol{r}) \in \boldsymbol{T}(\omega)\bigr\},$$

has one and only one solution $(\boldsymbol{c}^*, \boldsymbol{r}^*)$. Besides, $(\boldsymbol{c}^*, \boldsymbol{r}^*) = (\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}^*), \boldsymbol{\rho}(\dot{\boldsymbol{\eta}}^*))$, where $\dot{\boldsymbol{\eta}}^*$ is the unique solution to the 'classical' minimization problem $\inf_{\dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega)} j(\dot{\boldsymbol{\eta}})$ described in Section 1.

5. Concluding remarks

(a) While the original minimization problem is an *unconstrained* one over the space $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ with three unknowns (see Section 1), that found in Theorem 4.1 is a *constrained* minimization problem over the space $\boldsymbol{L}^2_{\mathrm{sym}}(\omega) \times \boldsymbol{L}^2_{\mathrm{sym}}(\omega)$ with six unknowns.

(b) In linear shell theory, the contravariant components of the *stress resultant* tensor field $(n^{\alpha\beta}) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega)$ and of the *bending moment* tensor field $(m^{\alpha\beta}) \in \boldsymbol{L}^2_{\mathrm{sym}}(\omega)$ are given in terms of the displacement vector field by $n^{\alpha\beta} = \varepsilon a^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $m^{\alpha\beta} = \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\boldsymbol{\eta})$, where the functions $a^{\alpha\beta\sigma\tau}$ are the contravariant components of the two-dimensional elasticity tensor of the shell. Since this tensor is uniformly positive definite, the above formulas are invertible and thus the minimization problem of Theorem 4.1 can be immediately recast as a *minimization problem with the stress resultants and bending moments as the primary unknowns*.

The soundness of the present approach is corroborated in the mechanics literature, where such an approach bears the befitting name of '*intrinsic equations of shell theory*'; in this direction, see the key paper of Opoka and Pietraszkiewicz [10].

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