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Differential Topology/Differential Geometry

The topology of corank 1 multi-singularities of stable smooth mappings of equidimensional manifolds

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Abstract

We study conditions for the coexistence of singularities of a stable smooth mapping of a closed manifold into a manifold of the same dimension n. Assuming that this mapping has only singularities of corank 1, we find universal linear relations between the Euler characteristics of the manifolds of multi-singularities in the image of the considered mapping. ***To cite this article: V.D. Sedykh, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

La topologie des multi-singularités de corank 1 des applications lisses et stables entre variétés de la même dimension. Nous étudions des conditions pour la co-existence de singularités d'une application lisse et stable d'une variété fermée dans une variété de la même dimension n. Sous l'hypothèse de que cette application a seulement des singularités de corank 1, nous obtenons relations linéaires universelles entre les nombres d'Euler des variétés de multi-singularités dans l'image de cette application. ***Pour citer cet article : V.D. Sedykh, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Let M and N be real C^∞-smooth n-dimensional manifolds, M be closed (compact without boundary). Consider a stable smooth mapping $f: M \to N$ (see [1]). We say that f is a *mapping of corank* $\leqslant 1$ if the dimension of the kernel of the derivative $f_{*x}: T_x M \to T_{f(x)} N$ does not exceed 1 for any $x \in M$.

Germs of f are classified with respect to the left-right equivalence (smooth transformations of local coordinates in N and M). The mapping f has a *singularity of type* A_μ at a given $x \in M$ if its local algebra at x is isomorphic to the $\mathbb{R}$-algebra $\mathbb{R}[[t]]/(t^{\mu+1})$ of truncated polynomials in one variable of degree at most μ. The mapping f of corank $\leqslant 1$ can have only singularities of types A_μ, where $0 \leqslant \mu \leqslant n$ (see [7]).

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The *multi-singularity* of f at $y \in N$ is the unordered set of singularities of f at points from $f^{-1}(y)$. Multi-singularities of the mapping f of corank $\leqslant 1$ are classified by elements $\mathcal{A} = A_{\mu_1} + \cdots + A_{\mu_p}$ of the free additive Abelian semigroup $\mathbb{A}$ whose generators are the symbols $A_0, A_1, \ldots$. The numbers $c(\mathcal{A}) = \mu_1 + \cdots + \mu_p$ and $d(\mathcal{A}) = c(\mathcal{A}) + p$ are called the *codimension* and the *degree* of a multi-singularity of type $\mathcal{A}$, respectively.

Remark 1. The mapping f has a multi-singularity of type 0 (the zero element of the semigroup $\mathbb{A}$) at any point of the complement $N \setminus f(M)$. If $\mathcal{A} = 0$, then $c(\mathcal{A}) = d(\mathcal{A}) = 0$.

The mapping f of corank $\leqslant 1$ can have only multi-singularities of codimension at most n. The set $\mathcal{A}_f$ of points $y \in N$, where f has a multi-singularity of type $\mathcal{A} \in \mathbb{A}$, is a smooth submanifold of codimension $c(\mathcal{A})$ in N. The Euler characteristic $\chi(\mathcal{A}_f)$ of $\mathcal{A}_f$ is the alternated sum of the Betti numbers of the homology groups with compact supports. Below, $\chi((k_1 A_{\mu_1} + \cdots + k_p A_{\mu_p})_f)$ is denoted by $\chi\binom{k_1,\ldots,k_p}{\mu_1,\ldots,\mu_p}$.

Notice that f is a finite mapping of a closed manifold. Therefore there exists the maximum of the number of points in $f^{-1}(y)$ by all $y \in N$. We will denote this maximum by d_f. It is clear that the mapping f of corank $\leqslant 1$ can have only multi-singularities of degree at most d_f.

Definition 1. The *index* $I_{\mathcal{A}}(X)$ of a multi-singularity of type $X = A_{\nu_1} + \cdots + A_{\nu_r}$ with respect to a multi-singularity of type $\mathcal{A} = A_{\mu_1} + \cdots + A_{\mu_p}$ is a nonnegative integer, calculated recursively as follows: 1) if $\mu^* = \max\{\mu_1, \ldots, \mu_p\} > \nu^* = \max\{\nu_1, \ldots, \nu_r\}$, then $I_{\mathcal{A}}(X) = 0$; 2) if $\mu^* \leqslant \nu^*$, then $I_{\mathcal{A}}(X) = \sum_{\nu_i = \mu^*} I_{\mathcal{A} - A_{\mu^*}}(X - A_{\nu_i}) + \sum_{\nu_i > \mu^*} I_{\mathcal{A} - A_{\mu^*}}(X - A_{\nu_i} + A_{\nu_i - \mu^* - 1})$, where $I_0(Y) = 1$ for any $Y \in \mathbb{A}$.

Consider the union $\mathcal{A}_f^\infty$ of the manifolds $(\mathcal{A} + kA_0)_f$ by all integers $k \geqslant 0$ (here, $(\mathcal{A} + kA_0)_f = \emptyset$ for any $k > d_f - d(\mathcal{A})$). By $\overline{\mathcal{A}_f^\infty}$ denote the closure of $\mathcal{A}_f^\infty$ in the ambient manifold N.

Theorem 2. *Let $\mathcal{A} \neq 0$ and the manifold $\mathcal{A}_f^\infty$ have odd dimension. Then the Euler characteristics $\chi(X_f)$ of the manifolds X_f of multi-singularities of types $X \in \mathbb{A}$ such that $X_f \subseteq \overline{\mathcal{A}_f^\infty}$ satisfy the relation*

$$\sum_X (-1)^{c(X)} I_{\mathcal{A}}(X) \chi(X_f) = 0. \tag{1}$$

If the manifold N is closed, then this formula is true for the case $\mathcal{A} = 0$ as well.

Corollary 3. *Let $f : M^n \to N^n$ be a stable smooth mapping of corank $\leqslant 1$. Assume that M is a closed manifold. Then the following statements hold.*

(1) *For any $\mathcal{A} \in \mathbb{A} \setminus \{0\}$ such that the manifold $\mathcal{A}_f$ of multi-singularities of type $\mathcal{A}$ of the mapping f has odd dimension, the Euler characteristic $\chi(\mathcal{A}_f)$ of the manifold $\mathcal{A}_f$ is a linear combination*

$$\chi(\mathcal{A}_f) = \sum_X K_{\mathcal{A}}(X) \chi(X_f) \tag{2}$$

(*with rational coefficients*) *of the Euler characteristics $\chi(X_f)$ of even-dimensional manifolds X_f of multi-singularities of types $X \in \mathbb{A}$, where $c(X) > c(\mathcal{A})$ and $d(X) \geqslant d(\mathcal{A})$. If the manifold N is closed, then this formula is true for the case $\mathcal{A} = 0$ as well.*

2) *The formula* (2) *is universal in the sense that every coefficient $K_{\mathcal{A}}(X)$ depends only on $\mathcal{A}$ and X* (*i.e. it does not depend on f and on the topology of the manifolds M, N*). *Namely,*

$$K_{\mathcal{A}}(X) = \sum_{i \geqslant 0} (-1)^i P_i(\mathcal{A}, X), \tag{3}$$

where $P_i(\mathcal{A}, X)$ is the sum of products of the form $\prod_{j=0}^{i} \frac{I_{Y_j}(Y_{j+1})}{I_{Y_j}(Y_j)}$ by all ordered sets $(Y_0, Y_1, \ldots, Y_{i+1})$ of elements of the semigroup $\mathbb{A}$ such that $Y_0 = \mathcal{A}, Y_{i+1} = X$, $d(\mathcal{A}) \leqslant d(Y_1) \leqslant \cdots \leqslant d(Y_i) \leqslant d(X)$, $c(\mathcal{A}) \leqslant c(Y_1) \leqslant \cdots \leqslant$

Table 1
List of formulas (2) for even $n \leqslant 4$

$c=1$	$2\chi\binom{1,k}{1,0} = 2\chi\binom{2,k-2}{1,0} + 2\chi\binom{2,k}{1,0} + 2\chi\binom{1,k-1}{2,0}$ $- 2\chi\binom{4,k-6}{1,0} - 6\chi\binom{4,k-4}{1,0} - 6\chi\binom{4,k-2}{1,0} - 2\chi\binom{4,k}{1,0} - 2\chi\binom{1,2,k-5}{2,1,0} - 4\chi\binom{1,2,k-3}{2,1,0} - 2\chi\binom{1,2,k-1}{2,1,0}$ $- 2\chi\binom{2,k-4}{2,0} - 2\chi\binom{2,k-2}{2,0} - 2\chi\binom{1,1,k-4}{3,1,0} - 3\chi\binom{1,1,k-2}{3,1,0} - 2\chi\binom{1,k-3}{4,0} - 2\chi\binom{1,k-1}{4,0}$
$c=3$	$2\chi\binom{3,k}{1,0} = 4\chi\binom{4,k-2}{1,0} + 4\chi\binom{4,k}{1,0} + 2\chi\binom{1,2,k-1}{2,1,0} + \chi\binom{1,1,k}{3,1,0}$
	$\chi\binom{1,1,k}{2,1,0} = \chi\binom{1,2,k-2}{2,1,0} + \chi\binom{1,2,k}{2,1,0} + 2\chi\binom{2,k-1}{2,0} + \chi\binom{1,1,k-1}{3,1,0} + \chi\binom{1,k}{4,0}$
	$2\chi\binom{1,k}{3,0} = \chi\binom{1,1,k-2}{3,1,0} + \chi\binom{1,1,k}{3,1,0} + 2\chi\binom{1,k-1}{4,0}$

$c(Y_i) < c(X)$, $c(Y_1) \equiv \cdots \equiv c(Y_i) \equiv c(\mathcal{A})$ (mod 2), and if $c(Y_{j+1}) = c(Y_j)$ for some $j = 0, \ldots, i-1$, then there is a positive integer $k \leqslant d(Y_{j+2}) - d(Y_j)$ such that $Y_{j+1} = Y_j + kA_0$.

(3) *For any $K_{\mathcal{A}}(X)$, there exists an integer $\alpha \geqslant 0$ such that $K_{\mathcal{A}}(X)2^{\alpha}$ is an integer. For any $\mathcal{A}, X \in \mathbb{A}$ such that $c(X) - c(\mathcal{A})$ is an odd positive integer, $K_{\mathcal{A}+kA_0}(X + kA_0) = K_{\mathcal{A}}(X)$ for every integer $k \geqslant 0$.*

(4) *The lists of formulas* (2) *for all possible $\mathcal{A} \in \mathbb{A}$ such that $c = c(\mathcal{A}) \leqslant 4$ are given in Table* 1 *for even $n \leqslant 4$ and in Table* 2 *for odd $n \leqslant 5$. In these tables, k is an arbitrary nonnegative integer* (*we let $\chi\binom{k_1,\ldots,k_p}{\mu_1,\ldots,\mu_p} = 0$ if there exists i such that $k_i < 0$*)*. The formula for $\chi\binom{k}{0}$ with $k = 0$ is valid if N is closed.*

(5) *If n is even, then the Euler characteristic $\chi(M)$ of M and the Euler characteristic $\chi(N)$ of N* (*in the case of a closed N*) *are linear combinations of the Euler characteristics $\chi(X_f)$ of even-dimensional manifolds X_f of multi-singularities of types $X \in \mathbb{A}$ of the mapping f. These combinations are universal in the same sense as above* (*their coefficients do not depend on f, M and N*)*. In particular, if $n \leqslant 4$, then*

$$\chi(M) = \sum_k \Big[k\chi\binom{k}{0} - (k+2)\chi\binom{2,k}{1,0} - \chi\binom{1,k}{2,0} + (5k+20)\chi\binom{4,k}{1,0} + (2k+9)\chi\binom{1,2,k}{2,1,0} + (k+4)\chi\binom{2,k}{2,0} + (k+5)\chi\binom{1,1,k}{3,1,0} + (k+3)\chi\binom{1,k}{4,0}\Big], \tag{4}$$

$$\chi(N) = \sum_k \Big[\chi\binom{k}{0} - \chi\binom{2,k}{1,0} + 5\chi\binom{4,k}{1,0} + 2\chi\binom{1,2,k}{2,1,0} + \chi\binom{2,k}{2,0} + \chi\binom{1,1,k}{3,1,0} + \chi\binom{1,k}{4,0}\Big]. \tag{5}$$

The sums in the formulas (4) *and* (5) *are taken by all nonnegative integers $k \leqslant d_f$* (*in the formula* (4) *for the case of a nonclosed N, we let $k\chi\binom{k}{0} = 0$ if $k = 0$*)*.*

Proof of the formula (1) is based on a resolution of multi-singularities of the mapping f which is similar to the resolution [9] of corank 1 singularities of the front of a stable Legendre mapping. This resolution is closely related to the iteration principle which is used in complex problems for the analysis of multiple-point cycles of generic holomorphic mappings having only singularities of corank 1 (see [2,4,5]).

Remark 2. The existence of relations of the form (2) (even for an arbitrary Whitney stratification of a smooth closed manifold) can be extracted from some papers on stratified sets (for example, from [8] or [6]). The formula (3), we obtained, supply a simple combinatorial algorithm for the calculation of all such relations between multi-singularities of a stable mapping $f : M^n \to N^n$ of corank $\leqslant 1$ for any fixed n.

Remark 3. It turns out that the system of relations of the form (2) is complete in the following sense. Let n and d be arbitrary fixed positive integers and $W_{n,d}$ be the class of all stable smooth mappings $f : M^n \to N^n$ of corank $\leqslant 1$, where M and N are any smooth closed manifolds of dimension n and $d_f \leqslant d$. Then any universal linear relation with real coefficients between the Euler characteristics of manifolds of multi-singularities of mappings $f \in W_{n,d}$ is a linear combination of the relations of the form (2) corresponding to $\mathcal{A} \in \mathbb{A}$ such that $c(\mathcal{A}) < n$, $c(\mathcal{A}) \equiv n - 1$ (mod 2) and $d(\mathcal{A}) \leqslant d$.

Table 2
List of formulas (2) for odd $n \leqslant 5$

$c=0$	$4\chi\binom{k}{0} = 2\chi\binom{1,k-2}{1,0} + 2\chi\binom{1,k}{1,0} - \chi\binom{3,k-6}{1,0} - 3\chi\binom{3,k-4}{1,0} - 3\chi\binom{3,k-2}{1,0} - \chi\binom{3,k}{1,0}$
	$- \chi\binom{1,1,k-5}{2,1,0} - 2\chi\binom{1,1,k-3}{2,1,0} - \chi\binom{1,1,k-1}{2,1,0} - \chi\binom{1,k-4}{3,0} - 2\chi\binom{1,k-2}{3,0} + \chi\binom{1,k}{3,0}$
	$+ 2\chi\binom{5,k-10}{1,0} + 10\chi\binom{5,k-8}{1,0} + 20\chi\binom{5,k-6}{1,0} + 20\chi\binom{5,k-4}{1,0} + 10\chi\binom{5,k-2}{1,0} + 2\chi\binom{5,k}{1,0} + 2\chi\binom{1,3,k-9}{2,1,0} + 8\chi\binom{1,3,k-7}{2,1,0}$
	$+ 12\chi\binom{1,3,k-5}{2,1,0} + 8\chi\binom{1,3,k-3}{2,1,0} + 2\chi\binom{1,3,k-1}{2,1,0} + 2\chi\binom{2,1,k-8}{2,1,0} + 6\chi\binom{2,1,k-6}{2,1,0} + 6\chi\binom{2,1,k-4}{2,1,0} + 2\chi\binom{2,1,k-2}{2,1,0}$
	$+ 2\chi\binom{1,2,k-8}{3,1,0} + 7\chi\binom{1,2,k-6}{3,1,0} + 8\chi\binom{1,2,k-4}{3,1,0} + 3\chi\binom{1,2,k-2}{3,1,0} + 2\chi\binom{1,1,k-7}{3,2,0} + 5\chi\binom{1,1,k-5}{3,2,0} + 3\chi\binom{1,1,k-3}{3,2,0}$
	$+ 2\chi\binom{1,1,k-7}{4,1,0} + 6\chi\binom{1,1,k-5}{4,1,0} + 5\chi\binom{1,1,k-3}{4,1,0} + \chi\binom{1,1,k-1}{4,1,0} + 2\chi\binom{1,k-6}{5,0} + 5\chi\binom{1,k-4}{5,0} + 2\chi\binom{1,k-2}{5,0} + \chi\binom{1,k}{5,0}$
$c=2$	$4\chi\binom{2,k}{1,0} = 6\chi\binom{3,k-2}{1,0} + 6\chi\binom{3,k}{1,0} + 4\chi\binom{1,1,k-1}{2,1,0} + 2\chi\binom{1,k}{3,0} - 10\chi\binom{5,k-6}{1,0} - 30\chi\binom{5,k-4}{1,0} - 30\chi\binom{5,k-2}{1,0} - 10\chi\binom{5,k}{1,0}$
	$- 9\chi\binom{1,3,k-5}{2,1,0} - 18\chi\binom{1,3,k-3}{2,1,0} - 9\chi\binom{1,3,k-1}{2,1,0} - 8\chi\binom{2,1,k-4}{2,1,0} - 8\chi\binom{2,1,k-2}{2,1,0} - 7\chi\binom{1,1,k-3}{3,2,0} - 3\chi\binom{1,1,k-1}{3,2,0}$
	$- 8\chi\binom{1,2,k-4}{3,1,0} - 12\chi\binom{1,2,k-2}{3,1,0} - 2\chi\binom{1,2,k}{3,1,0} - 7\chi\binom{1,1,k-3}{4,1,0} - 7\chi\binom{1,1,k-1}{4,1,0} - 6\chi\binom{1,k-2}{5,0} - 2\chi\binom{1,k}{5,0}$
	$4\chi\binom{1,k}{2,0} = 2\chi\binom{1,1,k-2}{2,1,0} + 2\chi\binom{1,1,k}{2,1,0} + 4\chi\binom{1,k-1}{3,0} - \chi\binom{1,3,k-6}{2,1,0} - 3\chi\binom{1,3,k-4}{2,1,0} - 3\chi\binom{1,3,k-2}{2,1,0} - \chi\binom{1,3,k}{2,1,0}$
	$- 2\chi\binom{2,1,k-5}{2,1,0} - 4\chi\binom{2,1,k-3}{2,1,0} - 2\chi\binom{2,1,k-1}{2,1,0} - 2\chi\binom{1,2,k-5}{3,1,0} - 4\chi\binom{1,2,k-3}{3,1,0} - 2\chi\binom{1,2,k-1}{3,1,0}$
	$- 3\chi\binom{1,1,k-4}{3,2,0} - 4\chi\binom{1,1,k-2}{3,2,0} + \chi\binom{1,1,k}{3,2,0} - 3\chi\binom{1,1,k-4}{4,1,0} - 4\chi\binom{1,1,k-2}{4,1,0} - \chi\binom{1,1,k}{4,1,0} - 4\chi\binom{1,k-3}{5,0} - 4\chi\binom{1,k-1}{5,0}$
$c=4$	$2\chi\binom{4,k}{1,0} = 5\chi\binom{5,k-2}{1,0} + 5\chi\binom{5,k}{1,0} + 2\chi\binom{1,3,k-1}{2,1,0} + \chi\binom{1,2,k}{3,1,0}$
	$2\chi\binom{1,2,k}{2,1,0} = 3\chi\binom{1,3,k-2}{2,1,0} + 3\chi\binom{1,3,k}{2,1,0} + 4\chi\binom{2,1,k-1}{2,1,0} + 2\chi\binom{1,2,k-1}{3,1,0} + \chi\binom{1,1,k}{3,2,0} + 2\chi\binom{1,1,k}{4,1,0}$
	$2\chi\binom{2,k}{2,0} = \chi\binom{2,1,k-2}{2,1,0} + \chi\binom{2,1,k}{2,1,0} + 2\chi\binom{1,1,k-1}{3,2,0} + \chi\binom{1,k}{5,0}$
	$\chi\binom{1,1,k}{3,1,0} = \chi\binom{1,2,k-2}{3,1,0} + \chi\binom{1,2,k}{3,1,0} + \chi\binom{1,1,k-1}{3,2,0} + \chi\binom{1,1,k-1}{4,1,0} + \chi\binom{1,k}{5,0}$
	$2\chi\binom{1,k}{4,0} = \chi\binom{1,1,k-2}{4,1,0} + \chi\binom{1,1,k}{4,1,0} + 2\chi\binom{1,k-1}{5,0}$

Remark 4. Similar results for Legendre mappings and for mappings from m-dimensional manifold into n-dimensional manifold where $m < n$ have been published in [9] and [10].

Remark 5. (Multi)singularities of mappings under consideration also satisfy coexistence conditions of different nature. In order to find them, different methods and objects are used: Thom polynomials, cobordism classes and so on (see [1,3]).

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