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Numerical Analysis

On the Hermite interpolation

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Abstract

In a given space of sufficiently differentiable functions, we show that the Hermite interpolation based on an arbitrary number of distinct points is possible if and only if it is possible when based on at most two distinct points. ***To cite this article: M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Sur l'interpolation d'Hermite. Si, dans un espace donné de fonctions suffisamment différentiables, tout problème d'interpolation d'Hermite impliquant au plus deux points distincts admet une solution unique, il en est de même de tout problème d'interpolation d'Hermite impliquant un nombre quelconque de points distincts. ***Pour citer cet article : M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Hermite interpolation

Throughout the Note, n denotes a fixed nonnegative integer and $\mathcal{E}$ a given $(n+1)$-dimensional space of C^n functions defined on a given real interval I with a nonempty interior. Given any integer r, $1 \leqslant r \leqslant n+1$, we use the expression *Hermite interpolation problem in $\mathcal{E}$* (*based on r distinct points*), for any problem of the following form:

$$\text{Find } U \in \mathcal{E} \text{ such that } U^{(j)}(\tau_i) = \alpha_{i,j}, \quad 1 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant \mu_i - 1,$$

in which $\tau_1, \ldots, \tau_r$ are pairwise distinct points in I, $\mu_1, \ldots, \mu_r$ are positive numbers such that $\sum_{i=1}^{r} \mu_i = n+1$, and $\alpha_{i,j}$, $1 \leqslant i \leqslant r$, $0 \leqslant j \leqslant \mu_i - 1$, are any real numbers. This includes *Taylor interpolation* (which corresponds to the particular case $r = 1$) as well as *Lagrange interpolation* (which corresponds to $r = n+1$).

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The result we present here is the following:

Theorem 1.1. *If any Hermite interpolation problem based on at most two distinct points has a unique solution in $\mathcal{E}$, then any Hermite interpolation problem has a unique solution in $\mathcal{E}$.*

2. Extended Chebyshev spaces

As is well-know, any Hermite interpolation problem based has a unique solution in the polynomial space $\mathcal{P}_n$ of degree n on $I := \mathbb{R}$. The underlying reason is that any nonzero polynomial of degree less than or equal to n has at most n zeros, counting multiplicities. Hermite interpolation is possible in other spaces, e.g. the space spanned by the functions $1, \cos x, \sin x$ on any interval I strictly contained in some $[a, a + 2\pi]$. Such spaces are natural generalisations of polynomial spaces see [1,3]).

Definition 2.1. The space $\mathcal{E}$ is said to be an *Extended Chebyshev space* (in short, EC-space) on I if any Hermite interpolation problem has a unique solution in $\mathcal{E}$.

How to check that a given space is an EC-space? Generalising the polynomial case, it is usual to characterise EC-spaces in terms of bounds of the total number of zeros (see (i) below) or in terms of regularity of certain *collocation matrices* (see (ii) below). To these two classical properties, the following proposition adds a less classical one, in terms of existence of certain bases with prescribed zeros.

Proposition 2.2. *Choosing a basis $(U_0, \ldots, U_n)$ in $\mathcal{E}$, let us set $\mathbf{U} := (U_0, \ldots, U_n)^{\mathrm{T}}$. Then, $\mathcal{E}$ is an EC-space on I if and only if it satisfies any of the following equivalent properties*:

- (i) *any nonzero element of $\mathcal{E}$ vanishes at most n times on I (with multiplicities)*;
- (ii) *for any $r \geqslant 1$, any positive numbers $\mu_1, \ldots, \mu_r$ with $\sum_{i=1}^{r} \mu_i = n + 1$, any pairwise distinct $\tau_1, \ldots, \tau_r \in I$, we have*
$$\det\bigl(\mathbf{U}(\tau_1), \ldots, \mathbf{U}^{(\mu_1-1)}(\tau_1), \ldots, \mathbf{U}(\tau_r), \ldots, \mathbf{U}^{(\mu_r-1)}(\tau_r)\bigr) \neq 0;$$
- (iii) *given any sequence $x_0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n < x_{n+1} \leqslant x_{2n} \leqslant x_{2n+1}$ in I, there exists a basis $(V_0, \ldots, V_n)$ of $\mathcal{E}$ such that, for $0 \leqslant i \leqslant n$, V_i vanishes (with multiplicities) on $(x_{i+1}, \ldots, x_{i+n})$, but it vanishes neither on $(x_i, \ldots, x_{i+n})$ nor on $(x_{i+1}, \ldots, x_{i+n+1})$ (see the meaning below).*

Properties (i) and (iii) require some explanation on the vocabulary we used. We first have to give sense to the count of multiples zeros for a function $U \in C^n(I)$. Given any $x \in I$, and any nonnegative integer $k \leqslant n + 1$, we say that *U vanishes k times at x* if $U(x) = \cdots = U^{(k-1)}(x) = 0$. In any subspace of $C^n(I)$, the multiplicity of a given zero is counted this way up to $n + 1$. On the other hand, given $k \leqslant n + 1$ non necessarily distinct $x_1, \ldots, x_k \in I$, among which exactly μ_i are equal to τ_i (with $\tau_1 < \cdots < \tau_r$ and with positive $\mu_1, \ldots, \mu_r$ such that $\mu_1 + \cdots + \mu_r = k$), we say that *$U$ vanishes (with multiplicities) on $(x_1, \ldots, x_k)$* if U vanishes μ_i times at τ_i for $1 \leqslant i \leqslant r$.

3. EC-spaces and W-spaces

As a particular case of Hermite interpolation, if $\mathcal{E}$ is an EC-space, Taylor interpolation is always possible. More generally:

Definition 3.1. The space $\mathcal{E}$ is said to be a *W-space* on I if any Taylor interpolation problem has a unique solution in $\mathcal{E}$.

In case $k \leqslant n$, we say that *$U \in \mathcal{E}$ vanishes exactly k times at x* if $U(x) = \cdots = U^{(k-1)}(x) = 0$ and $U^{(k)}(x) \neq 0$. Similarly to Proposition 2.2, we can characterise W-spaces as follows.

Proposition 3.2. *The space $\mathcal{E}$ is a W-space on I if and only if it satisfies any of the following equivalent requirements*:

(i)′ *for any $x \in I$, any nonzero element of $\mathcal{E}$ vanishes at most n times at x;*
(ii)′ *the Wronskian of $(U_0, \dots, U_n)$ never vanishes on I, i.e.,*

$$W(U_0, \dots, U_n)(x) := \det\bigl(\mathbf{U}(x), \dots, \mathbf{U}^{(n)}(x)\bigr) \neq 0, \quad x \in I;$$

(iii)′ *for any $x \in I$, there exists a basis $(V_0, \dots, V_n)$ of $\mathcal{E}$ such that, for $0 \leqslant i \leqslant n$, V_i vanishes exactly i times at x.*

In Proposition 2.2 as well as in Proposition 3.2, the second property is often the practical way to check whether a given space is, or is not, an EC-space or a W-space. Proving (ii)′ is obviously easier than proving (ii), and of course (ii)′ can be satisfied while (ii) is not. For instance the Wronskian of the three functions $1, \cos x, \sin x$, never vanishes on $\mathbb{R}$ although the space they span is an EC-space only on any interval strictly contained in some $[a, a + 2\pi]$.

Below we recall a classical result which establishes a strong interesting link between EC-spaces and W-spaces. It says that, in theory, instead of proving (ii), it may be possible to limit ourselves to Wronskians in the following sense.

Theorem 3.3. *Suppose the existence of a nested sequence of W-spaces on I*

$$\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \mathcal{E}_{n-1} \subset \mathcal{E}_n = \mathcal{E},$$

the space $\mathcal{E}_k$ being $(k+1)$-dimensional for $0 \leqslant k \leqslant n$. Then $\mathcal{E}$ is an EC-space on I.

Note that the converse property is true when the interval I is closed and bounded. The latter result means that Hermite interpolation is possible in the space $\mathcal{E}$ as soon as Taylor interpolation is possible in any subspace of some nested sequence of subspaces. The simplest illustration of Theorem 3.3 is the polynomial case, with the nested sequence $\mathcal{E}_i := \mathcal{P}_i$. Beyond the simple framework of polynomials, Theorem 3.3 states a crucial and nontrivial theoretical result. If, in practice, it does not provide direct help to show that a given space is, or is not, an EC-space, it will be one of the key-points in the proof of Theorem 1.1.

4. Proof of Theorem 1.1

By analogy with the Bernstein polynomials $B_i^n(x) := \binom{n}{i}(1-x)^{n-i}x^i$, $0 \leqslant i \leqslant n$, let us first introduce the following definition.

Definition 4.1. Given $a, b \in I$, with $a \neq b$, we say that a basis $(\mathcal{B}_0, \dots, \mathcal{B}_n)$ of $\mathcal{E}$ is *a Bernstein-like basis relative to* (a, b) if, for $0 \leqslant i \leqslant n$, $\mathcal{B}_i$ vanishes exactly i times at a and exactly $(n-i)$ times at b.

Clearly, $(\mathcal{B}_0, \dots, \mathcal{B}_n)$ is a Bernstein-like basis relative to (a, b) if and only if $(\mathcal{B}_n, \dots, \mathcal{B}_0)$ is a Bernstein-like basis relative to (b, a). On the other hand, if $a < b$, $(\mathcal{B}_0, \dots, \mathcal{B}_n)$ is a Bernstein-like basis relative to (a, b) if and only if the functions $V_i := \mathcal{B}_{n-i}$, $0 \leqslant i \leqslant n$, satisfy the properties required in (iii) of Proposition 2.2 relative to the particular sequence $x_0 := \cdots := x_n := a$, $x_{n+1} := \cdots := x_{2n+1} := b$.

Bernstein-like bases are basic tools to prove Theorem 1.1. Indeed, modelled on Propositions 2.2 and 3.2, we can state the following result.

Proposition 4.2. *Any Hermite interpolation problem based on at most two distinct points has a unique solution in $\mathcal{E}$ if and only if one of the following equivalent statement is satisfied.*

(i)″ *given any distinct $a, b \in I$, any nonzero element of $\mathcal{E}$ vanishes at most n times on the set $\{a, b\}$;*
(ii)″ *given any distinct $a, b \in I$ and any nonnegative integers i, j such that $i + j = n + 1$, we have*

$$\det\bigl(\mathbf{U}(a), \dots, \mathbf{U}^{(i-1)}(a), \mathbf{U}(b), \dots, \mathbf{U}^{(j-1)}(b)\bigr) \neq 0;$$

(iii)″ *$\mathcal{E}$ possesses a Bernstein-like basis relative to any pair of distinct points of I.*

In other words, Theorem 1.1 says that, in any of the properties (i)–(iii) of Proposition 2.2, instead of considering an arbitrary number ($\leqslant n+1$) of pairwise distinct points of I, we can limit ourselves to at most two arbitrary distinct points of I. The practical interest of Theorem 1.1 is clear: obviously, proving (ii)″ is significantly easier than proving (ii).

Sketch of the proof of Theorem 1.1. We suppose that $\mathcal{E}$ satisfies (i)″–(iii)″, and we want to prove that $\mathcal{E}$ is an EC-space on I. Here are the main steps.

1) Prove that $\mathcal{E}$ is an EC-space on any interval J with a nonempty interior assumed to be strictly contained in I.

2) In case $I=[\alpha,\beta]$, prove that $\mathcal{E}$ can be extended into an $(n+1)$-dimensional space $\mathcal{E}_1$ of C^n functions on some interval $I_1=[\alpha_1,\beta_1]$, with $\alpha_1<\alpha$, $\beta_1>\beta$, so that $\mathcal{E}_1$ satisfies (ii)″ on I_1.

3) On account of part 1), the proof is complete if I is open or half-open. If I is a closed bounded interval $[\alpha,\beta]$, $\mathcal{E}$ is proved to be an EC-space on I by applying part 1) to the extension $\mathcal{E}_1$ constructed in part 2).

Proof of part 1. Select $a\in J$, $b\in I\setminus J$, and let $(B_0,\dots,B_n)$ be a Bernstein-like basis relative to (a,b). For $0\leqslant k\leqslant n$, let $\mathcal{E}_k$ denote the $(k+1)$-dimensional space spanned by $B_0,\dots,B_k$. According to Theorem 3.3 and to (i)′, it is sufficient to check that any nonzero element $U\in\mathcal{E}_k$ vanishes at most k times at any given point $x\in J$. From (i)″, we know that U vanishes at most n times on $\{x,b\}$, and from the definition of a Bernstein-like basis, we know that it vanishes at least $n-k$ times at b.

Proof of part 2. Suppose that $I=[\alpha,\beta]$. With no loss of generality, one can assume each function $U_0,\dots,U_n$ to be defined and C^n on $\mathbb{R}$. For a given integer i, $1\leqslant i\leqslant n+1$, consider the function

$$\Psi_i(x,y):=\det\bigl(\mathbf{U}(x),\dots,\mathbf{U}^{(i-1)}(x),\mathbf{U}(y),\dots,\mathbf{U}^{(j-1)}(y)\bigr),\quad x,y\in\mathbb{R},$$

with $j:=n+1-i$. From (ii)″ we know that

$$\Psi_i(x,y)\neq 0\quad\text{for all } x,y\in I,\ x\neq y. \tag{1}$$

One can prove that (see [2, Lemma 3.1])

$$\Psi_i(x,y)=(x-y)^{ij}\widetilde{\Psi}_i(x,y),\quad x,y\in\mathbb{R}, \tag{2}$$

where the function $\widetilde{\Psi}_i$ is continuous on I^2, and satisfies

$$\widetilde{\Psi}_i(x,x)=\frac{1!2!\cdots(i-1)!\,1!2!\cdots(j-1)!}{1!2!\cdots n!}\det\bigl(\mathbf{U}(x),\dots,\mathbf{U}^{(n)}(x)\bigr),\quad x\in I. \tag{3}$$

Using (1), (2), and (3), one can see that $\widetilde{\Psi}_i$ does not vanish on the whole of I^2. We can thus find $\varepsilon_i>0$ such that $\widetilde{\Psi}_i$ does not vanish on $[\alpha-\varepsilon_i,\beta+\varepsilon_i]^2$. Since there are only a finite number of possible choices for i, we can find a positive ε so that (ii)″ is still satisfied for any pair (a,b) of distinct points of $I_1:=[\alpha-\varepsilon,\beta+\varepsilon]$.

Detailed proofs can be found in [2].

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