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C. R. Acad. Sci. Paris, Ser. I 340 (2005) 135–140
COMPTES RENDUS MATHEMATIQUE
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Algebraic Geometry

Rings of invariants for representations of quivers

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Received 4 June 2004; accepted after revision 7 December 2004

Available online 30 December 2004

Presented by Jean-Pierre Demailly

Abstract

In this Note we compute the generators of the ring of invariants for quiver factorization problems, generalizing results of Le Bruyn and Procesi. In particular, we find a necessary and sufficient combinatorial criterion for the projectivity of the associated invariant quotients. Further, we show that the non-projective quotients admit open immersions into projective varieties, which still arise from suitable quiver factorization problems. ***To cite this article: M. Halic, M.S. Stupariu, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Anneaux d'invariants des représentations de carquois. Dans cette Note nous calculons les générateurs des anneaux d'invariants pour certains problèmes de factorisation associés aux représentations de carquois, généralisant un résultat démontré par Le Bruyn et Procesi. En particulier, nous déduisons un critère combinatoire nécéssaire et suffisant pour la projectivité du quotient. En plus, nous démontrons que les quotients non-projectifs peuvent être immergés de manière ouverte dans varietés projectives qui proviennent elles mêmes de problèmes de factorisation de carquois appropriés. ***Pour citer cet article : M. Halic, M.S. Stupariu, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Considérons un carquois $Q = (Q_0, Q_1, h, t)$, un sous-ensemble de sommets $S \subseteq Q_0$ (qu'on appelle *marqués*) et une représentation $r = (V, \psi)$ de Q, où $V = \{V_q\}_{q \in Q_0}$ est une famille d'espaces vectoriels de dimension finie, sur un corps algébriquement clos k de caractéristique nulle. Nous étudions l'action naturelle du groupe réstreint $G_S := \prod_{q \in S} \mathrm{Gl}(V_q)$ sur l'espace de représentation W_r. Dans le Théorème 1.1, nous calculons les générateurs de l'anneau des invariants $k[W_r]^{G_S}$: notamment nous démontrons que cet anneau est engendré par les traces des cycles marqués et par les fonctions de coordonnées des trajectoires orientées, dont la source et le but ne sont pas

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.12.012

marqués. Ceci généralise un résultat dû à Le Bruyn et Procesi qui traite le cas où tous les sommets sont marqués. Comme application, nous trouvons un critère combinatoire nécessaire et suffisant, impliquant Q et S seulement, qui assure la projectivité des quotients invariants $W_r/\!\!/(G_S,\chi)$, où χ est un caractère de G_S.

Une question naturelle qui se pose est celle de la complétion des quotients non-projectifs $W_r/\!\!/(G_S,\chi)$. Nous introduisons un procédé combinatiore qu'on appelle *d'élargissement*, par lequel on ajoute des sommets et des arêts à un carquois. Ce procédé détruit les cycles marqués et les trajectoires orientées de Q, dont la source et le but ne sont pas marqués. Le résultat principal dans cette direction est formulé dans le Théorème 1.3, qui dit que pour chaque carquois Q avec des sommets marqués S on trouve un élargissement $(\widetilde{Q},\widetilde{S})$ ayant la propriété suivante : pour chaque charactère χ de G_S et pour chaque représentation r de Q, il y a un charactère $\tilde{\chi}$ de $G_{\widetilde{S}}$ et une représentation $\tilde{r}$ de $\widetilde{Q}$, tel que $W_{\tilde{r}}/\!\!/(G_{\widetilde{S}},\tilde{\chi})$ est projective et contient $W_r/\!\!/(G_S,\chi)$ comme ouvert de Zariski. Ceci traduit dans le langage de la géométrie invariante et généralise le résultat de [3], qui porte sur la théorie du contrôle des systèmes linéaires.

1. Introduction

A *quiver* Q is a quartet (Q_0,Q_1,h,t) consisting of the sets Q_0 and Q_1 of vertices, respectively arrows, and the maps $h,t:Q_1\to Q_0$, which associate to every arrow a the *head*, respectively the *tail* of a. Throughout this note we will consider only finite quivers, i.e. we will assume that the sets Q_0 and Q_1 are finite. If $a_1,\dots,a_n$ are arrows such that $h(a_i)=t(a_{i+1})$ for any $i=1,\dots,n-1$, they give rise to the *oriented path* $a_n\cdots a_1$. If, moreover, one has $h(a_n)=t(a_1)$, this oriented path will be called a *cycle*. A vertex q is called a *source* (respectively a *sink*) if all the arrows meeting q are starting from q (respectively directed to q). Any quiver without cycles has at least one source and one sink.

Let further k be an algebraically closed field of characteristic 0. A *representation* $r=(V,\psi)$ of a quiver Q (over the field k) is given by a family of finite dimensional k-vector spaces $(V_q)_{q\in Q_0}$ and a family of linear maps $(\psi_a)_{a\in Q_1}$, with $\psi_a:V_{t(a)}\to V_{h(a)}$ for any arrow a.

Let now $r=(V,\psi)$ be a fixed representation of a quiver Q of dimension vector α. There is a natural action of the group $\prod_{q\in Q_0}\mathrm{Gl}(V_q)$ on the *representation space* $W_r:=\bigoplus_{a\in Q_1}\mathrm{Hom}(V_{t(a)},V_{h(a)})$, given by

$$(g_q)_q\times(\varphi_a)_a:=(g_{h(a)}\circ\varphi_a\circ g_{t(a)}^{-1})_a.$$

In many concrete situations one is interested only in the action of a smaller symmetry group (see for instance [4, Section 1.5] for a symplectic approach to the problem). Consequently, for a subset of vertices $S\subset Q_0$, we define the group $G_S:=\prod_{q\in S}\mathrm{Gl}(V_q)\subset\prod_{q\in Q_0}\mathrm{Gl}(V_q)$, and consider its induced action on the space W_r; we denote by $\rho_{r,S}$ the corresponding representation. This data defines a *quiver factorization problem* associated to the combinatorial data (Q,S). In the special case when $S=Q_0$, it will be called a *standard quiver factorization problem*. For $S\subset Q_0$ as above, we will call the vertices in S *marked*, and those in $Q_0\setminus S$ *unmarked*.

The structure of the ring of invariants $k[W_r]^{G_{Q_0}}$ in the case when all the vertices are marked is described by Le Bruyn and Procesi in [1, Theorem 1]; namely this ring is generated by traces of oriented cycles in the quiver Q. Our aim is to generalize this result for quivers with both marked and unmarked vertices.

Theorem 1.1. *Let Q be a quiver with marked vertices $S\subset Q_0$, and let r be a representation of Q with representation space W_r. Then the ring of invariants $k[W_r]^{G_S}$ is generated by traces of marked cycles and by coordinate functions of oriented paths with unmarked source and sink.*

For a character χ of the group G_S, the associated invariant quotient

$$W_r/\!\!/(G_S,\chi)=\mathrm{Proj}\Big(\bigoplus_{n\geqslant 0}k[W_r]^{G_S}_{\chi^n}\Big)$$

will be called a *QFP-quotient*. Since $W_r/\!\!/(G_S,\chi)$ is projective over $\mathrm{Spec}(k[W_r]^{G_S})$, we deduce that

Corollary 1.2. *The following statements are equivalent*:

(i) $W_r /\!\!/ (G_S, \chi)$ *is a projective variety, for every character* χ *of* G_S;
(ii) $k[W_r]^{G_S} = k$;
(iii) Q *contains no marked cycles and no oriented paths with unmarked source and sink.*

We introduce a combinatorial construction, that we call *enlargement*, which enables us to construct completions of QFP-quotients which are still QFP-quotients.

Theorem 1.3. *For a quiver* (Q, S) *with marked vertices, there is an enlargement* $(\widetilde{Q}, \widetilde{S})$ *with the property that for any character* $\chi \in \mathcal{X}^*(G_S)$ *and any representation* r *of* Q, *there exist a character* $\tilde{\chi} \in \mathcal{X}^*(G_{\widetilde{S}})$ *and a representation* $\tilde{r}$ *of* $\widetilde{Q}$ *such that* $W_{\tilde{r}} /\!\!/ (G_{\widetilde{S}}, \tilde{\chi})$ *is projective and contains* $W_r /\!\!/ (G_S, \chi)$ *as a Zariski open subset.*

2. Computation of the ring of invariants

For proving Theorem 1.1, we modify the initial quiver and its representation, without changing the ring of invariants, such that after the modification the result of [1] can be applied. This will be done in two steps: first we 'cut' the initial quiver into several pieces, at the unmarked vertices, where the symmetry group does not act, and secondly we 'collapse' the unmarked vertices of each of the connected sub-quivers which have been obtained at the first step.

Cutting procedure. Let $m \in Q_0 \setminus S$ be an unmarked vertex which is neither a source nor a sink. The cutting Q^m of Q at m is constructed as follows: the set of vertices is $Q_0^m := (Q_0 \setminus \{m\}) \cup \{m', m''\}$. The set of arrows Q_1^m is obtained from Q_1 in the following way:

– any loop at m gives rise to an arrow from m' to m'';
– any incoming arrow in m (not a loop) yields an arrow having the same origin, but pointed to m'';
– any outgoing arrow from m (not a loop) yields an arrow having the same head, but starting from m';
– all other arrows remain unchanged.

We keep the set $S \subset Q_0^m$ untouched; this makes sense, since we chose m to be unmarked. The representation r of Q induces in a natural fashion a representation r^m of Q^m: the vector spaces corresponding to m' and m'' are both equal to V_m, whereas the other remain unchanged.

Clearly, the representation spaces W_r and W_{r^m} coincide as G_S*-modules*, and therefore the rings of invariants coincide too. The main remark is that after a finite number of cuttings, we obtain a (possibly disconnected) quiver with the property that all its unmarked vertices are either sources or sinks. Consequently, it is enough to prove the main result for quivers which fulfill this property.

Collapsing procedure. Assume now that (Q, S) is a connected quiver with marked vertices, such that all its unmarked vertices are either sources or sinks, and consider a representation r of Q. We construct a new quiver $\widehat{Q}$, depending on both the pair (Q, S) and the dimension vector of r, whose set of vertices is given by $\widehat{Q}_0 := S \cup \{\widehat{m}\}$, that is we replace all the unmarked vertices of Q with a single vertex $\widehat{m}$. The arrows of $\widehat{Q}$ are obtained in the following way:

– the arrows between marked vertices remain unchanged;
– any arrow in Q with an unmarked tail q for which $\dim(V_q) = d$ is replaced with d arrows with tail $\widehat{m}$ and with the same head;
– any arrow in Q with an unmarked head q for which $\dim(V_q) = d$ is replaced with d arrows with head $\widehat{m}$ and with the same tail.

The effect of replacing an unmarked vertex $q \in Q_0$ of dimension d and an arrow meeting this vertex with the vertex $\widehat{m}$ of dimension one together with d (incoming or outgoing) arrows is that of choosing a basis in V_q.

We consider the representation $\hat{r}$ of $\widehat{Q}$ for which the vector space associated to $\widehat{m}$ is equal to $V_{\widehat{m}} := k$, whereas the other ones coincide with those corresponding to r. The representation spaces W_r and $W_{\hat{r}}$ are *isomorphic* as G_S*-modules*, so that the rings of invariants $k[W_r]^{G_S}$ and $k[W_{\hat{r}}]^{G_S}$ are still isomorphic. The main remark is that the action of the one-parameter subgroup $\lambda : k^\times \to G_S,\ \lambda(t) := (t^{-1}\,\mathrm{Id}_{V_q})_{q \in S}$ coincides with the natural $\mathrm{Gl}(V_{\widehat{m}})$-action on the space $W_{\hat{r}}$. Since $G_S \times \mathrm{Gl}(V_{\widehat{m}}) = G_{\widehat{Q}_0}$, we conclude that any G_S-invariant function on $W_{\hat{r}}$ is also $G_{\widehat{Q}_0}$-invariant, that is the rings $k[W_{\hat{r}}]^{G_S}$ and $k[W_{\hat{r}}]^{G_{\hat{Q}_0}}$ coincide. Now we apply the result of [1], and deduce that the generators of the latter one are traces of cycles. But a cycle in $\widehat{Q}$ is either a cycle in Q through *marked* vertices or a cycle through $\widehat{m}$, which corresponds to an oriented path in Q whose source and sink are unmarked. Since the dimension of $\widehat{m}$ equals one, a cycle through $\widehat{m}$ has only one trace, which corresponds to a coordinate function of the associated path in Q.

3. Completions of the QFP-quotients

Enlargement procedure. Let $m \in Q_0$ be a vertex. The *enlargement* $\widetilde{Q}^m$ *of* Q *at* m is constructed as follows: the vertices of $\widetilde{Q}^m$ coincide with the vertices of the quiver Q^m, described at the cutting procedure, while the set of arrows is defined as $\widetilde{Q}_1^m := Q_1^m \cup \{b\}$, where b is an arrow from m' to m''. The set of marked vertices is obtained as follows:

– if m is unmarked, we put $\widetilde{S}^m := S \cup \{m''\}$;
– if m is marked, we put $\widetilde{S}^m := (S \setminus \{m\}) \cup \{m', m''\}$.

A representation r of Q induces the representation $\tilde{r}^m$ of $\widetilde{Q}^m$ for which the vector spaces corresponding to m' and m'' are both equal to V_m, while the other remain unchanged. The b-component of the corresponding family $\tilde{\psi}^m$ of linear maps is the identity of $\mathrm{End}(V_m)$, and the rest are the natural ones.

An *enlargement* $(\widetilde{Q}, \widetilde{S})$ of (Q, S) is obtained applying a finite number of times the procedure above. Our discussion shows that a representation r of Q naturally induces a representation $\tilde{r}$ of $\widetilde{Q}$.

Remark 1. The effect of the enlargement procedure is that of destroying the marked cycles and the oriented paths with unmarked source and sink, without introducing new ones. Indeed, after enlarging Q at m, all the cycles through m disappear. On the other hand, if m is an unmarked vertex, by enlarging Q at m, the oriented paths ending in m yield oriented paths in $\widetilde{Q}^m$ with one unmarked vertex less.

Now we are going to compare the representation spaces when we perform an enlargement at one vertex. First, we observe that the representation space corresponding to r decomposes as

$$\begin{aligned} W_r &= \bigoplus_{a \notin t^{-1}(m) \cup h^{-1}(m)} \mathrm{Hom}\big(V_{t(a)}, V_{h(a)}\big) \oplus \bigoplus_{a \in t^{-1}(m) \setminus h^{-1}(m)} \mathrm{Hom}\big(V_m, V_{h(a)}\big) \oplus \bigoplus_{a \in h^{-1}(m) \setminus t^{-1}(m)} \mathrm{Hom}\big(V_{t(a)}, V_m\big) \oplus \bigoplus_{a \in h^{-1}(m) \cap t^{-1}(m)} \mathrm{End}(V_m) \\ &=: W_r^0 \oplus W_r^- \oplus W_r^+ \oplus W_r^{\circlearrowleft}. \end{aligned}$$

Regardless whether $m \in Q_0$ is marked or not, the underlying vector space of the representation space for $\widetilde{Q}^m$ is $W_{\tilde{r}^m} = \mathrm{End}(V_m) \oplus W_r$, and the symmetry group is $G_{\widetilde{S}^m} = \mathrm{Gl}(V_m) \times G_S$. What distinguishes between the unmarked and the marked cases is the representation $G_{\widetilde{S}^m} \to \mathrm{Gl}(W_{\tilde{r}^m})$. Namely, define

$$\vartheta_m : \mathrm{Gl}(V_m) \longrightarrow \mathrm{Gl}(W_r), \quad \vartheta_m(\gamma)(w) := (w^0, w^-, \gamma w^+, \gamma w^{\circlearrowleft}),$$

for $w = (w^0, w^-, w^+, w^{\circlearrowleft}) \in W_r$, and

$$\delta : G_S \longrightarrow G_{\widetilde{S}^m}, \quad \delta(g) := \begin{cases} (1, g) & \text{if } m \text{ is unmarked,} \\ (g_m, g) & \text{if } m \text{ is marked.} \end{cases}$$

Some straightforward computations show:

– In $\mathrm{Gl}(W_r)$, the following commutation relations hold:

$$\begin{cases} \vartheta_m(\gamma)\rho_{r,S}(g) = \rho_{r,S}(g)\vartheta_m(\gamma) & \text{if } m \text{ is unmarked,} \\ \vartheta_m(g_m\gamma g_m^{-1})\rho_{r,S}(g) = \rho_{r,S}(g)\vartheta_m(\gamma) & \text{if } m \text{ is marked.} \end{cases} \tag{1}$$

– The representation $\rho_{\tilde{r}^m,\widetilde{S}^m} : G_{\widetilde{S}^m} \to \mathrm{Gl}(W_{\tilde{r}^m})$ is given by

$$\rho_{\tilde{r}^m,\widetilde{S}^m}(\gamma, g)(u, w) = \begin{cases} \big(\gamma u, \vartheta_m(\gamma)\rho_{r,S}(g)w\big) & \text{if } m \text{ is unmarked,} \\ \big(\gamma u g_m^{-1}, \vartheta_m(\gamma g_m^{-1})\rho_{r,S}(g)w\big) & \text{if } m \text{ is marked,} \end{cases}$$

for all $\widetilde{w} := (u, w) \in \mathrm{End}(V_m) \oplus W_r = W_{\tilde{r}^m}$.

– Under the natural identification $\mathrm{Gl}(V_m) \cong \mathrm{Gl}(V_m) \times \{1\} \subset G_{\widetilde{S}^m}$,
  – if m is unmarked, $G_{\widetilde{S}^m} = \mathrm{Gl}(V_m) \times \delta(G_S)$;
  – if m is marked, $G_{\widetilde{S}^m} = \mathrm{Gl}(V_m) \rtimes \delta(G_S)$, where $\delta(G_S)$ acts on $\mathrm{Gl}(V_m)$ by conjugation.

– The inclusion $\iota_1 : W_r \cong \{1\} \times W_r \hookrightarrow W_{\tilde{r}^m}$ is equivariant with respect to the G_S-action on W_r and the $\delta(G_S)$-action on $W_{\tilde{r}^m}$.

With these preparations we can state the main result of this section.

Proposition 3.1. *Let (Q, S) be a quiver with marked vertices, $m \in Q_0$ a vertex, and r a representation of Q. Consider the enlargement $(\widetilde{Q}^m, \widetilde{S}^m)$ of Q at m, and the representation $\tilde{r}^m$ of $\widetilde{Q}^m$. Then, for each character $\chi \in \mathcal{X}^*(G_S)$, one finds a character $\tilde{\chi} \in \mathcal{X}^*(G_{\widetilde{S}^m})$ which fulfills the following conditions*:

(i) $\iota_1(W_r^{\mathrm{ss}}(G_S, \chi)) = \iota_1(W_r) \cap W_{\tilde{r}^m}^{\mathrm{ss}}(G_{\widetilde{S}^m}, \tilde{\chi})$;
(ii) *the natural morphism $W_r /\!\!/ (G_S, \chi) \to W_{\tilde{r}^m} /\!\!/ (G_{\widetilde{S}^m}, \tilde{\chi}^m)$ is an open immersion.*

Proof. The idea is to construct invariant functions on $W_{\tilde{r}^m}$ out of invariant functions on W_r. For a regular function $f \in k[W_r]$, we define $\varphi_f \in k[\mathrm{Gl}(V_m) \times W_r] = k[W_{\tilde{r}^m}]_{(\det_m)}$ by the formula $\varphi_f(\gamma, w) := f(\vartheta_m(\gamma^{-1})w)$, where $\det_m(\gamma, w) := \det(\gamma)$. It follows that for a suitably large positive integer N (depending on f), the function $\tilde{f}_N := (\det_m)^N \varphi_f$ is regular on $W_{\tilde{r}^m}$, and satisfies the equality $\tilde{f}_N(\gamma, w) = \det^N(\gamma) \cdot f(\vartheta_m(\gamma^{-1})w)$, $\forall (\gamma, w) \in \mathrm{Gl}(V_m) \times W_r$. In particular, $\tilde{f}_N(1, w) = f(w)$, for all $w \in W_r$. Assume now that we start with a function $f \in k[W_r]^{G_S}_{\chi^\ell}$, that is $f(\rho_{r,S}(g)w) = \chi^\ell(g) \cdot f(w)$, for all $g \in G_S$ and $w \in W_r$. Using the relations (1), one checks that $\tilde{f}_N$ satisfies the equality $\tilde{f}_N(\rho_{\tilde{r}^m,\widetilde{S}^m}(\gamma, g)(u, w)) = \tilde{\chi}_{\ell,N}(\gamma, g)\tilde{f}_N(u, w)$ for all $(u, w) \in W_{\tilde{r}^m}$, where

$$\tilde{\chi}_{\ell,N}(\gamma, g) := \begin{cases} \chi^\ell(g) \cdot \det^N(\gamma) & \text{if } m \text{ is unmarked,} \\ \chi^\ell(g) \cdot \det^N(\gamma g_m^{-1}) & \text{if } m \text{ is marked.} \end{cases}$$

This means that $\tilde{f}_N \in k[W_{\tilde{r}^m}]^{G_{\tilde{S}^m}}_{\tilde{\chi}_{\ell,N}}$.

Consider now an integer $\ell > 0$ such that there is a *finite* set of functions $\{f_j\}_{j\in J} \subset k[W_r]^{G_S}_{\chi^\ell}$ with the property that $\bigcup_{j\in J}\{f_j \neq 0\} = W_r^{\mathrm{ss}}(G_S, \chi)$. We choose the integer N large enough, that all the $\tilde{f}_{j,N}$'s are regular on $W_{\tilde{r}^m}$. Since the inclusion ι_1 is equivariant, it follows that $\iota_1(W_r^{\mathrm{ss}}(G_S, \chi)) \subset \iota_1(W_r) \cap W_{\tilde{r}^m}^{\mathrm{ss}}(G_{\widetilde{S}^m}, \tilde{\chi}_{\ell,N})$. The other inclusion is obvious, and this proves the first part of the proposition.

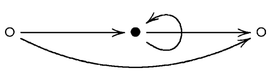
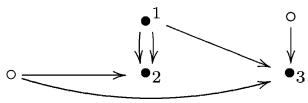
For the second part, we observe that in both the unmarked and the marked cases, the orbit $\rho_{\tilde{r}^m,\widetilde{S}^m}(G_{\widetilde{S}^m}) \cdot \iota_1(W_r^{\mathrm{ss}}(G_S, \chi)) = \mathrm{Gl}(V_m) \times \Omega$, with $\Omega := \vartheta_m(\mathrm{Gl}(V_m)) \cdot W_r^{\mathrm{ss}}(G_S, \chi)$, is a $G_{\widetilde{S}^m}$-invariant open subset of $W_{\tilde{r}^m}$, which is acted on freely by $\mathrm{Gl}(V_m)$. Notice that for unmarked m, Ω actually coincides with $W_r^{\mathrm{ss}}(G_S, \chi)$. The morphism $\mathrm{Gl}(V_m) \times \Omega \to \Omega$, $(u, w) \mapsto \vartheta_m(u^{-1})w$ is $\mathrm{Gl}(V_m)$-invariant, so it descends to the quotient $(\mathrm{Gl}(V_m) \times \Omega)/\mathrm{Gl}(V_m) \to \Omega$. One immediately sees that this application is an isomorphism. Moreover, the composition

$W_r^{\mathrm{ss}}(G_S,\chi) \cong \iota_1(W_r^{\mathrm{ss}}(G_S,\chi)) \to \mathrm{Gl}(V_m) \times \Omega \to \Omega$ is the natural inclusion $W_r^{\mathrm{ss}}(G_S,\chi) \subset \Omega$, which is an open immersion. The second statement follows immediately from the properties of the categorical quotient. □

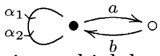
Moreover, one can prove that the main result of [5] still holds for actions of reductive groups on linear spaces, so that only finitely many open subsets of $W_{\tilde{r}^m}$ can be realized as the semi-stable locus corresponding to a character of $G_{\widetilde{S}^m}$. Consequently, there is a constant $C_{r,\chi} > 0$ having the property that for all integers $\ell, N > 0$, with $N/\ell > C_{r,\chi}$, the semi-stable loci $W_{\tilde{r}^m}^{\mathrm{ss}}(G_{\widetilde{S}^m}, \tilde{\chi}_{\ell,N})$ coincide.

We claim that the proposition and the remark above imply Theorem 1.3. Indeed, after a finite number of steps one obtains an enlargement $(\widetilde{Q}, \widetilde{S})$ without marked cycles, and any oriented path has at most one unmarked vertex. Hence, by corollary 1.2, any associated QFP-quotient is projective. We observe that different choices of enlargements typically lead to different, birational, completions of the initial QFP-quotient. Moreover, the number of completions obtained in this way is finite, up to isomorphism.

Example 1. Helmke compactifies in [3] the space of regular, controllable linear systems by generalized, controllable linear systems. These spaces can be interpreted as QFP-quotients associated to the following quivers, where the marked vertices are represented by bullets:

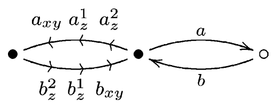
Q:  $\widetilde{Q}$: 

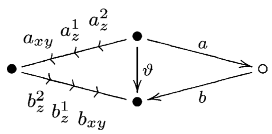
The corresponding characters are $\chi(g) := \det(g)$ and $\tilde{\chi}(g_1, g_2, g_3) := \det^{1-N}(g_1)\det^N(g_2)\det^M(g_3)$, with $N, M > 0$ suitably large.

Example 2. Donaldson shows in [2] that the framed $\mathrm{SU}(\ell)$-instantons on S^4 can be understood in terms of monads on $\mathbb{CP}^2$. Their moduli space can be identified with the QFP-quotient of the quiver Q_0: 

corresponding to the trivial character. Applying the enlargement procedure directly, one is led to a quiver which has no interpretation in terms of monads. We remedy this by noticing that there is an equivariant morphism from the representation space of Q to that of Q_0, and that the representation space of its enlargement $\widetilde{Q}$ admits a natural monad-theoretical interpretation.

$$\alpha_1 := b_{xy}a_z^1 = b_z^2 a_{xy},$$
$$\alpha_2 := b_{xy}a_z^2 = -b_z^1 a_{xy}.$$

Q: 

$\widetilde{Q}$: 

It would be interesting to relate this compactification with that of Maruyama, which uses semi-stable, torsion-free sheaves.

Acknowledgements

Both authors have been partially supported by: EAGER-European Algebraic Geometry Research Training Network, contract No. HPRN-CT-2000-00099 (BBW).

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