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Mathematical Analysis

A Lidskii type formula for Dixmier traces

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Abstract

We present a new formula to compute Dixmier traces $\tau_\omega(T)$ of pseudodifferential operators (respectively, almost periodic pseudodifferential operators) of order $-n$ on n-dimensional compact Riemannian manifolds (respectively, $\mathbb{R}^n$). Under a natural condition on the operator T, we show that $\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_{\lambda\notin \frac{1}{t}G} \lambda \,\mathrm{d}\mu_T(\lambda)$, where G is any bounded neighborhood of $0 \in \mathbb{C}$ and μ_T is the Brown spectral measure of T. If T is measurable, then the ω-limit may be replaced with the true (ordinary) limit. Our approach works equally well in both type I and II settings. ***To cite this article: N.A. Azamov, F.A. Sukochev, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Une nouvelle formule pour calculer les traces de Dixmier. Nous présentons une nouvelle formule pour calculer les traces de Dixmier $\tau_\omega(T)$ des opérateurs pseudodifférentiels (respectivement, des opérateurs pseudodifférentiels presque périodiques) d'ordre $-n$ sur des variétés compactes de dimension n (respectivement, $\mathbb{R}^n$). Lorsque T satisfait une condition naturelle, nous montrons que $\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_{\lambda\notin \frac{1}{t}G} \lambda \,\mathrm{d}\mu_T(\lambda)$, où G est un voisinage borné de 0 dans $\mathbb{C}$ et μ_T est la mesure spectrale de Brown de T. Si T est mesurable, on peut remplacer la limite faible par la limite au sens usuel. Notre approche s'applique aux types I et II. ***Pour citer cet article : N.A. Azamov, F.A. Sukochev, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

D'après le théorème de Lidskii (sous la forme générale semi-finie de [1]), si $\mathcal{N}$ est un facteur de Von Neumann semi-fini muni d'une trace fidèle, normale, semi-finie, la trace $\tau(T)$ d'un opérateur $T \in L^1(\mathcal{N}, \tau)$ est $\tau(T) = \int_{\sigma(T)\setminus\{0\}} \lambda \,\mathrm{d}\mu_T(\lambda)$, où μ_T est la mesure spectrale de Brown de T. Lorsque $\mathcal{N}$ est un facteur de type I (respectivement, lorsque T est un opérateur normal) μ_T est la mesure de comptage sur l'ensemble $\{\lambda_n(T)\}_{n=1}^{\infty}$ des valeurs propres de T (respectivement, la mesure τ-spectrale de T, donnée par $\mu_T(B) = \tau(\chi_B(T))$ sur

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les boréliens $B \subset \mathbb{C}$). Nous donnons ici une formule analogue pour les traces de Dixmier. Nous définissons $\mathcal{L}^{(1,\infty)} = \{T \in \mathcal{N}\colon \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\,\mathrm{d}s < \infty\}$, où $\mu_t(T)$ est la t-ième valeur singulière de T [10]. Lorsque ω est un état sur $L^\infty(0,\infty)$, s'annulant sur les fonctions à support compact et satisfaisant $\omega(Mf) = \omega(f)$ pour toute $f \in L^\infty(0,\infty)$, avec $M(f)(t) = \frac{1}{\log(1+t)} \int_0^t f(s)\frac{\mathrm{d}s}{s}$, $t \in (0,\infty)$, la trace de Dixmier [5,8] $\tau_\omega(T)$ est $\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\,\mathrm{d}s$ pour $T \in \mathcal{L}^{(1,\infty)}$, positif et s'étend par linéarité dans le cas général.

Dans le cas classique, si T est un opérateur à trace, compact, auto-adjoint, le théorème de Lidskii se déduit directement du théorème spectral pour les opérateurs positifs et de la convergence absolue de la série des valeurs propres de T. Si $T = T^*$ est dans $\mathcal{L}^{(1,\infty)}$, cette série peut diverger, ce qui est un obstacle non trivial. En effet, nous montrerons comme cas particulier de notre résultat principal que si $T = T^* \in \mathcal{L}^{(1,\infty)}$ vérifie $|\lambda_n(T)| \leqslant \frac{C}{n}$ pour un $C > 0$ et tout $n \geqslant 1$, avec $|\lambda_1(T)| \geqslant |\lambda_2(T)| \geqslant \cdots$ si τ_ω est une trace de Dixmier, alors $\tau_\omega(T) = \omega\text{-}\lim_{N\to\infty} \frac{1}{\log(1+N)} \sum_{n=1}^N \lambda_n(T)$. De plus, si T est mesurable au sens de Connes [5], on peut remplacer dans cette formule la limite faible par la limite au sens usuel.

1. Introduction

The well-known Lidskii theorem (in its general semifinite form given in [1]) asserts that if $\mathcal{N}$ is a semifinite von Neumann factor with a faithful normal semifinite trace τ, then the trace $\tau(T)$ of an arbitrary operator $T \in L^1(\mathcal{N},\tau)$ is given by $\tau(T) = \int_{\sigma(T)\setminus\{0\}} \lambda\,\mathrm{d}\mu_T(\lambda)$, where μ_T is the Brown's measure of T. In the case when $\mathcal{N}$ is a type I factor, the measure μ_T is the counting measure on the set of all eigenvalues of T. In this paper, we present an analogue of such a formula for Dixmier traces. Let ω be a state on $L^\infty(0,\infty)$ which vanishes on functions with compact support and such that $\omega(Mf) = \omega(f)$ for every $f \in L^\infty(0,\infty)$, where $M(f)(t) = \frac{1}{\log(1+t)} \int_0^t f(s)\frac{\mathrm{d}s}{s}$. It will be convenient to write $\omega\text{-}\lim_{t\to\infty} f(t)$ instead of $\omega(f)$, $f \in L^\infty(0,\infty)$. The Dixmier trace [5,6] $\tau_\omega(T)$ defined on the ideal $\mathcal{L}^{(1,\infty)} := \{T \in \mathcal{N}\colon \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\,\mathrm{d}s < \infty\}$ is given by $\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\,\mathrm{d}s$ if $T \in \mathcal{L}^{(1,\infty)}$ is positive and by linearity otherwise. Here, $\mu_t(T) = \inf\{\|TE\|\colon E$ is a projection in $\mathcal{N}$ with $\tau(1-E) \leqslant t\}$ is the tth generalized s-number of the operator T.

In the case of a standard (normal) trace, the assertion of the Lidskii theorem for self-adjoint operators is immediate due to the absolute convergence of the series $\sum_{n\geqslant 1} \lambda_n(T)$ of any $T = T^*$ from the trace class. This is not the case any longer for Dixmier (non-normal) traces, since the latter series diverges for any $T = T^* \in \mathcal{L}^{(1,\infty)}$ which does not belong to the trace class.

The distribution function of $T \in \mathcal{N}$ is defined by $\lambda_t(T) := \tau(\chi_{(t,\infty)}(|T|))$, $t > 0$. We have $\mu_s(T) = \inf\{t \geqslant 0\colon \lambda_t(T) \leqslant s\}$ and for any $s,t > 0$, $s \geqslant \lambda_t(T)$ if and only if $\mu_s(T) \leqslant t$. Furthermore, $\int_0^{\lambda_t(T)} \mu_s(T)\,\mathrm{d}s = \tau(|T|\chi_{(t,\infty)}(|T|))$, $\forall t > 0$. These facts may be found in [2,10]. We write $S \prec\!\prec T$ iff $\int_0^t \mu_s(S)\,\mathrm{d}s \leqslant \int_0^t \mu_s(T)\,\mathrm{d}s$, $\forall t > 0$.

Our main result is given in Theorem 2.11 below. The Lidskii type formula given there holds for all operators $T \in \mathcal{L}^{(1,\infty)}$ satisfying $\mu_t(T) \leqslant C/t$ for some $C > 0$ and all $t > 0$. The set of such operators form an ideal in $\mathcal{N}$ denoted by $\mathcal{L}^{1,\mathrm{w}}$. The ideal $\mathcal{L}^{1,\mathrm{w}}$ usually arises in geometric applications. In particular, if $\mathcal{N}$ is the algebra of all bounded operators on $L^2(M)$ where M is a compact Riemannian n-manifold (respectively, if $\mathcal{N}$ is the II_∞-factor $L^\infty(\mathbb{R}^n) \rtimes \mathbb{R}^n_{\mathrm{discr}}$ [4,12]), the ideal $\mathcal{L}^{1,\mathrm{w}}$ contains all pseudodifferential operators (respectively, all almost periodic pseudodifferential operators) of order $-n$.

An operator T from $\mathcal{L}^{(1,\infty)}$ is said to be measurable if $\tau_\omega(T)$ does not depend on the state ω [5]. For an arbitrary subset $A \subseteq \mathcal{N}$, we denote by A_m the set of measurable elements from A.

Our formula takes an especially simple form for the case of measurable operators T. In this case, $\tau_\omega(T)$ coincides with the true limit $\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_{\lambda\notin\frac{1}{t}G} \lambda\,\mathrm{d}\mu_T(\lambda)$ for an arbitrary ω.

Our results depend crucially on the recent characterization of positive measurable operators from $\mathcal{L}^{(1,\infty)}$ as those for which the limit $\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\,\mathrm{d}s$ exists [11, Theorem 6.6], and the spectral characterization of sums of commutators in type II factors [8,9].

2. Lidskii formulae for Dixmier traces

Lemma 2.1. *If $T \geqslant 0$ in $\mathcal{L}^{1,\mathrm{w}}$ then there exists $S \in \mathcal{L}^{1,\mathrm{w}}_m$ such that $T \leqslant S$, $\operatorname{supp}(S) \leqslant \operatorname{supp}(T)$ and $ST = TS$.*

Proof. If $\mathcal{N}$ is a type II factor, then the assertion follows from [7, Proposition 3.2]. The type I case is straightforward. □

The proof of the following lemma is straightforward and is therefore omitted.

Lemma 2.2. *If $T \in \mathcal{L}^{(1,\infty)}_m$ then T^*, $\operatorname{Re}(T)$, $\operatorname{Im}(T) \in \mathcal{L}^{(1,\infty)}_m$. The same assertion also holds for $\mathcal{L}^{1,\mathrm{w}}$.*

Remark 1. The positive and negative parts of a measurable self-adjoint operator $T \in \mathcal{L}^{(1,\infty)}$ are not necessarily measurable.

Lemma 2.3 (see [2]). *For $T \in \mathcal{L}^{(1,\infty)}$, we have $\lambda_{1/t}(T) \leqslant Ct\log t$, for some $C > 0$, and all sufficiently large t.*

For brevity, we write $f_t(T) = \int_0^t \mu_s(T)\,\mathrm{d}s$ and $g_t(T) = \int_0^{\lambda_{1/t}(T)} \mu_s(T)\,\mathrm{d}s$, $t > 0$. The results given in Proposition 2.4 and Lemma 2.5 below are similar to those obtained in [2, Proposition 2.4] under different assumptions on ω and T.

Proposition 2.4. *If $T \geqslant 0$ in $\mathcal{L}^{1,\mathrm{w}}$, then for every $C > 0$*

$$\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} f_t(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^{Ct\log t} \mu_s(T)\,\mathrm{d}s$$

and if one of the ω-limits is a true limit then so is the other.

Proof. For $T \geqslant 0$ in $\mathcal{L}^{1,\mathrm{w}}$, we have $\frac{1}{\log(1+t)}\left|\int_0^{Ct\log t} \mu_s(T)\,\mathrm{d}s - f_t(T)\right| \leqslant \frac{M}{\log(1+t)}(\log(Ct\log t) - \log t) \to 0$ as $t \to \infty$ for some $M > 0$. The second assertion is proved in [2, Proposition 2.4]. □

Lemma 2.5. *If $T \geqslant 0$ in $\mathcal{L}^{1,\mathrm{w}}$ then*

$$\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} g_t(T). \tag{1}$$

If T is measurable then the ω-limit can be replaced with the true limit.

Proof. It follows from Lemma 2.3 and Proposition 2.4 that

$$\omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} g_t(T) \leqslant \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^{Ct\log t} \mu_s(T)\,\mathrm{d}s = \tau_\omega(T). \tag{2}$$

Further, since $s > \lambda_{1/t}$ implies $\mu_s(T) \leqslant 1/t$, we have $f_t(T) \leqslant g_t(T) + \frac{1}{t}(t - \lambda_{1/t}(T)) \leqslant g_t(T) + 1$. Hence,

$$\tau_\omega(T) \leqslant \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} g_t(T). \tag{3}$$

Combining (2) and (3), we arrive at (1). The second assertion follows from Proposition 2.4 and [11, Theorem 6.6]. □

Lemma 2.6. *If $A, B, C \geqslant 0$ in $\mathcal{L}^{1,\mathrm{w}}$ and $C = A + B$ then*

$$\text{(i)}\ \lim_{t\to\infty}\frac{|f_t(A)+f_t(B)-f_t(C)|}{\log(1+t)}=0,\qquad \text{(ii)}\ \lim_{t\to\infty}\frac{|g_t(A)+g_t(B)-g_t(C)|}{\log(1+t)}=0. \tag{4}$$

Proof. Let $A' \geqslant 0$ and $B' \geqslant 0$ be operators from $\mathcal{L}^{(1,\infty)}$ such that $\mu_t(A') = \mu_t(A)$, $\mu_t(B') = \mu_t(B)$ $\forall t > 0$ and $A'B' = 0$. Let $C' = A' + B'$. According to [3, Lemma 2.3] we have $C' \prec\prec C$. Combining this fact with the observation that $\mu(C) \prec\prec \mu(A) + \mu(B)$ [10, Proposition 2.4], we see that it is sufficient to prove that

$$\lim_{t\to\infty}\frac{1}{\log(1+t)}\left|f_t(A')+f_t(B')-f_t(C')\right|=0. \tag{5}$$

Since A' and B' are orthogonal, we have

$$g_t(A')+g_t(B')-g_t(C')=0. \tag{6}$$

Combining (6) with Lemma 2.3 and the argument in the proof of Proposition 2.4 we arrive at (4(i)). The proof of (4(ii)) is similar. □

Lemma 2.7. *Let $T \in \mathcal{L}^{1,\mathrm{w}}$ be normal and $T = T_1 - T_2 + \mathrm{i}T_3 - \mathrm{i}T_4$, where $T_1,\dots,T_4 \geqslant 0$. Then $\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty}\frac{1}{\log(1+t)}(g_t(T_1) - g_t(T_2) + \mathrm{i}g_t(T_3) - \mathrm{i}g_t(T_4))$. If T is measurable then the ω-limit can be replaced with the true limit.*

Proof. The first assertion follows from Lemma 2.5 and the linearity of Dixmier traces. Let T be measurable and self-adjoint. Let $S \geqslant 0$ in $\mathcal{L}^{(1,\infty)}$ be a measurable operator commuting with T_- such that $S - T_- \geqslant 0$ and $\operatorname{supp}(S) \leqslant \operatorname{supp}(T_-)$ (see Lemma 2.1). We have $\tau_\omega(T) = \tau_\omega(S+T) - \tau_\omega(S) = \omega\text{-}\lim_{t\to\infty}\frac{1}{\log(1+t)}f_t(S+T) - \omega\text{-}\lim_{t\to\infty}\frac{1}{\log(1+t)}f_t(S)$. Since $0 \leqslant T, T+S$ are measurable, we may combine [11, Theorem 6.6] with Lemma 2.5 to obtain $\tau_\omega(T) = \lim_{t\to\infty}\frac{1}{\log(1+t)}g_t(S+T) - \lim_{t\to\infty}\frac{1}{\log(1+t)}g_t(S)$. Since $S+T = S-T_-+T_+$ and the operators $S - T_-$ and T_+ are disjoint, we have $g_t(S+T) = g_t(S-T_-) + g_t(T_+)$. Eq. (4(ii)) of Lemma 2.6 with $A = T_-$, $B = S - T_-$ and $C = S$ now yields $\lim_{t\to\infty}\frac{1}{\log(1+t)}(g_t(T_+) - g_t(T_-) - g_t(S+T) + g_t(S)) = 0$. Taking the ω-limit, we conclude from [11, Theorem 6.6] that

$$\begin{aligned}\tau_\omega(T) &= \omega\text{-}\lim_{t\to\infty}\frac{g_t(T_+)-g_t(T_-)}{\log(1+t)} = \omega\text{-}\lim_{t\to\infty}\frac{g_t(S+T)-g_t(S)}{\log(1+t)}\\ &= \lim_{t\to\infty}\frac{g_t(S+T)-g_t(S)}{\log(1+t)} = \lim_{t\to\infty}\frac{g_t(T_+)-g_t(T_-)}{\log(1+t)}.\end{aligned} \tag{7}$$

If T is normal, the assertion now follows from Lemma 2.2. □

Lemma 2.8. *If $T \in \mathcal{L}^{1,\mathrm{w}}$ is normal and $a > 0$ then $\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty}\frac{1}{\log(1+t)}\int_{\lambda\notin Q_t}\lambda\,\mathrm{d}\mu_T(\lambda)$, where $Q_t = \{x + \mathrm{i}y \in \mathbb{C}\colon |tx| \leqslant a, |ty| \leqslant a\}$ $\forall t > 0$. If T is measurable, then the ω-limit can be replaced with the true limit.*

Proof. We may take $a = 1$, by dilation invariance of ω [5, p. 305]. Let $T = T_1 - T_2 + \mathrm{i}T_3 - \mathrm{i}T_4$, where $T_1,\dots,T_4 \geqslant 0$. For $T \geqslant 0$ in $\mathcal{L}^{1,\mathrm{w}}$, we have $g_t(T) = \int_{1/t}^{\infty}\lambda\,\mathrm{d}\mu_T(\lambda)$ (see, e.g. [2]). Let $\bar{A}$ be the complement of $A \subseteq \mathbb{C}$, $R_t := \{\lambda \in \mathbb{C}\colon |\mathrm{Re}(\lambda)| \leqslant 1/t\}$ and $I_t := \{\lambda \in \mathbb{C}\colon |\mathrm{Im}(\lambda)| \leqslant 1/t\}$. For any Borel set $B \subseteq \mathbb{R}$, we have $\int_B \lambda\,\mathrm{d}\mu_{\mathrm{Re}(T)}(\lambda) = \int_{\{\lambda:\ \mathrm{Re}(\lambda)\in B\}}\mathrm{Re}(\lambda)\,\mathrm{d}\mu_T(\lambda)$, $\int_B \lambda\,\mathrm{d}\mu_{\mathrm{Im}(T)}(\lambda) = \int_{\{\lambda:\ \mathrm{Im}(\lambda)\in B\}}\mathrm{Im}(\lambda)\,\mathrm{d}\mu_T(\lambda)$ and so

$$\begin{aligned}\int_{\bar{Q}_t}\lambda\,\mathrm{d}\mu_T(\lambda) &= \int_{\bar{Q}_t}\mathrm{Re}(\lambda)\,\mathrm{d}\mu_T(\lambda) + \mathrm{i}\int_{\bar{Q}_t}\mathrm{Im}(\lambda)\,\mathrm{d}\mu_T(\lambda)\\ &= \int_{\bar{R}_t}\mathrm{Re}(\lambda)\,\mathrm{d}\mu_T(\lambda) + \int_{\bar{Q}_t\cap R_t}\mathrm{Re}(\lambda)\,\mathrm{d}\mu_T(\lambda) + \mathrm{i}\int_{\bar{I}_t}\mathrm{Im}(\lambda)\,\mathrm{d}\mu_T(\lambda) + \mathrm{i}\int_{\bar{Q}_t\cap I_t}\mathrm{Im}(\lambda)\,\mathrm{d}\mu_T(\lambda)\end{aligned}$$

$$= \int\limits_{\{|\xi|>1/t\}} \xi \,\mathrm{d}\mu_{\mathrm{Re}(T)}(\xi) + \int\limits_{\bar{Q}_t \cap R_t} \mathrm{Re}(\lambda)\,\mathrm{d}\mu_T(\lambda) + \mathrm{i} \int\limits_{\{|\xi|>1/t\}} \xi \,\mathrm{d}\mu_{\mathrm{Im}(T)}(\xi) + \mathrm{i} \int\limits_{\bar{Q}_t \cap I_t} \mathrm{Im}(\lambda)\,\mathrm{d}\mu_T(\lambda).$$

The sum of the first and the third terms in the expression above gives $\tau_\omega(T)$ after taking the ω-limit with respect to $t \to \infty$ (see Lemma 2.7). We shall now show that $\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_{\bar{Q}_t \cap R_t} \mathrm{Re}(\lambda)\,\mathrm{d}\mu_T(\lambda) = 0$ and $\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_{\bar{Q}_t \cap R_t} \mathrm{Im}(\lambda)\,\mathrm{d}\mu_T(\lambda) = 0$. We prove the first equality, the second is proved analogously. In fact, it suffices to prove that $\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_{\{\mathrm{Im}(\lambda)>1/t\}\cap R_t} \mathrm{Re}(\lambda)\,\mathrm{d}\mu_T(\lambda) = 0$. We have

$$\Bigg| \int\limits_{\{\mathrm{Im}(\lambda)>1/t\}\cap R_t} \mathrm{Re}(\lambda)\,\mathrm{d}\mu_T(\lambda) \Bigg| \leqslant \frac{1}{t} \int\limits_{\{\mathrm{Im}(\lambda)>1/t\}\cap R_t} \mathrm{d}\mu_T(\lambda) \leqslant \frac{1}{t} \int\limits_{\{\mathrm{Im}(\lambda)>1/t\}} \mathrm{d}\mu_T(\lambda) = \frac{1}{t} \int\limits_{1/t}^{\infty} \mathrm{d}\mu_{\mathrm{Im}(T)}(\xi)$$
$$= \frac{1}{t} \tau\big(\chi_{(1/t,\infty)}(T_3)\big) = \frac{1}{t} \lambda_{1/t}(T_3) \leqslant C.$$

The last inequality follows from the equivalence of $\mu_{Ct}(T_3) \leqslant 1/t$ and $\lambda_{1/t}(T_3) \leqslant Ct$. For the case of measurable T, the proof is the same. □

Lemma 2.9. *Let T be a normal operator from $\mathcal{L}^{1,\mathrm{w}}$ and let G be a bounded Borel neighborhood of $0 \in \mathbb{C}$. Then $\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_{\lambda \notin \frac{1}{t}G} \lambda \,\mathrm{d}\mu_T(\lambda)$. If T is measurable, then the ω-limit can be replaced with the true limit.*

Proof. For an arbitrary bounded neighborhood G of $0 \in \mathbb{C}$ there exist squares Q_a and Q_b such that $Q_a \subseteq G \subseteq Q_b$. Hence, Lemma 2.8 implies that it is sufficient to prove that

$$\lim_{t\to\infty} \frac{1}{\log(1+t)} \int\limits_{Q_{t/b}\setminus Q_{t/a}} |\lambda|\,\mathrm{d}\mu_T(\lambda) = 0. \tag{8}$$

The set $Q_{t/b} \setminus Q_{t/a}$ consists of four trapeziums and it is suffices to prove the above limit for one of them, $D_t := \{z \in Q_{t/b} \setminus Q_{t/a}\colon \mathrm{Re}(tz) \in [a,b]\}$, for example. We have

$$\frac{1}{2} \int\limits_{D_t} |\lambda|\,\mathrm{d}\mu_T(\lambda) \leqslant \int\limits_{D_t} \mathrm{Re}(\lambda)\,\mathrm{d}\mu_T(\lambda) \leqslant \int\limits_{t\,\mathrm{Re}(\lambda)\in[a,b]} \mathrm{Re}(\lambda)\,\mathrm{d}\mu_T(\lambda) = \int\limits_{a/t}^{b/t} \lambda \,\mathrm{d}\mu_{\mathrm{Re}(T)}(\lambda) = \int\limits_0^{\lambda_{a/t}} \mu_s \,\mathrm{d}s - \int\limits_0^{\lambda_{b/t}} \mu_s \,\mathrm{d}s.$$

By Lemma 2.5, we can replace upper limits $\lambda_{a/t}$ and $\lambda_{b/t}$ by t/a and t/b respectively. Then $\int_0^{t/a} \mu_s \,\mathrm{d}s - \int_0^{t/b} \mu_s \,\mathrm{d}s \leqslant \int_{t/b}^{t/a} C/s \,\mathrm{d}s \leqslant C \log \frac{b}{a}$. □

The following lemma follows from [8].

Lemma 2.10. *If $S \in \mathcal{L}^{1,\mathrm{w}}$ then there exists a normal operator $T \in \mathcal{L}^{(1,\infty)}$ such that the Brown spectral measures of S and T coincide and $\tau_\omega(S) = \tau_\omega(T)$.*

The proof of the following theorem (which is the main result of this note) follows from Lemmas 2.9 and 2.10.

Theorem 2.11. *If $S \in \mathcal{L}^{1,\mathrm{w}}$ and G is a bounded Borel neighborhood of $0 \in \mathbb{C}$, then $\tau_\omega(S) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \times \int_{\lambda \notin \frac{1}{t}G} \lambda \,\mathrm{d}\mu_S(\lambda)$. If S is measurable, then the ω-limit can be replaced with the true limit.*

We specialize the result above to the case when $\mathcal{N}$ is an infinite-dimensional factor of type I_∞.

Corollary 2.12. *Let T be a compact operator on an infinite-dimensional Hilbert space $\mathcal{H}$ such that $\mu_n(T) \leqslant C/n,\ n \geqslant 1$, for some $C > 0$. If $\lambda_1, \lambda_2, \ldots$ is the list of eigenvalues of T counting the multiplicities such that $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots$, then*

$$\mathrm{Tr}_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \sum_{\lambda\in\sigma(T),\ \lambda\notin\frac{1}{t}G} \lambda\mu_T(\lambda) = \omega\text{-}\lim_{N\to\infty} \frac{1}{\log N} \sum_{i=1}^{N} \lambda_i,$$

where $\mu_T(\lambda)$ is the algebraic multiplicity of the eigenvalue λ. If T is measurable then the ω-limit can be replaced with the true limit.

Proof. The first equality is an immediate consequence of Theorem 2.11. By Lemma 2.10, it is sufficient to prove the second equality for a normal operator T. Let $G := \{z \in \mathbb{C}:\ |z| < 1\}$. It is enough to show that $\sum_{k\in A_N\cup B_N} |\lambda_k| <$ const, where $A_N = \{k \in \mathbb{N}:\ k \leqslant N, |\lambda_k| \leqslant 1/N\}$ and $B_N = \{k \in \mathbb{N}:\ k > N, |\lambda_k| > 1/N\}$. We have, $\sum_{k\in A_N} |\lambda_k| \leqslant 1$. That $\sum_{k\in B_N} |\lambda_k|$ is bounded follows from the condition $|\lambda_k| < C/k,\ k \in \mathbb{N}$, for some $C > 0$ and estimate (8). □

The following corollary follows from the combination of Corollary 2.12 and [5, Proposition IV.2.5].

Corollary 2.13 [9, Proposition 1]. *If M is a compact Riemannian n-manifold and T is a pseudodifferential operator of order $-n$ on M, then* $\mathrm{Tr}_\omega(T) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{k=1}^{N} \lambda_k$.

Let $\mathcal{N} = L^\infty(\mathbb{R}^n) \rtimes \mathbb{R}^n_{\mathrm{discr}}$ and let $T^\sharp$ be (the image of) an almost periodic pseudodifferential operator of order $-n$ (see, for example, [12]). Then $T^\sharp \in \mathcal{L}^{1,\mathrm{w}}$.

Corollary 2.14. $\tau_\omega(T^\sharp) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_{|\lambda|>1/t} \lambda \, \mathrm{d}\mu_T(\lambda)$.

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