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Partial Differential Equations

On Pfaff systems with L^p coefficients in dimension two

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Abstract

We prove that the Cauchy problem associated with a Pfaff system with coefficients in L^p_{loc}, $p > 2$, in a connected and simply-connected open subset Ω of $\mathbb{R}^2$ has a unique solution provided that its coefficients satisfies a compatibility condition in the distributional sense. ***To cite this article: S. Mardare, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Sur les systèmes de Pfaff en dimension deux. On montre que le problème de Cauchy associé à un système de Pfaff avec des coefficients dans L^p_{loc}, $p > 2$, dans un ouvert connexe et simplement connexe Ω de $\mathbb{R}^2$ admet une solution unique pourvu que ses coefficients satisfassent une condition de compatibilité au sens des distributions. ***Pour citer cet article : S. Mardare, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Les notations sont définies dans la version anglaise. Soit Ω un ouvert connexe et simplement connexe de $\mathbb{R}^2$, soit x_0 un point de Ω, et soit Y^0 une matrice de $\mathbb{M}^{q\times\ell}$. Il est alors bien connu (voir, e.g., Thomas [7]) que le système de Pfaff

$$\partial_i Y = Y A_i \quad \text{dans } \Omega,\ i = 1, 2,$$
$$Y(x^0) = Y^0,$$

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admet une solution unique $Y \in \mathcal{C}^2(\Omega; \mathbb{M}^{q\times\ell})$ si les coefficients A_i appartiennent à l'espace $\mathcal{C}^1(\Omega; \mathbb{M}^\ell)$ et satisfont la condition de compatibilité

$$\partial_1 A_2 + A_1 A_2 = \partial_2 A_1 + A_2 A_1 \quad \text{dans } \Omega. \tag{1}$$

L'objet de cette Note est d'établir que ce résultat reste vrai sous les hypothèses affaiblies que les coefficients A_i appartiennent à l'espace $L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$, $p > 2$, la condition de compatibilité ci-dessus étant alors satisfaite au sens des distributions (voir Théorème 3.2 dans la version anglaise). La preuve repose sur deux résultats principaux : un résultat de stabilité pour les systèmes de Pfaff à coefficients dans $L^p(\Omega)$ établi dans le Théorème 2.1 et un résultat d'approximation, sous la contrainte non linéaire (1), des champs de matrices A_i établi dans le Lemme 3.1 de la version anglaise.

La démonstration complète de ces résultats, esquissée dans la version anglaise, se trouve dans [5].

1. Preliminaries

The notations $\mathbb{M}^{q\times\ell}, \mathbb{M}^\ell, \mathbb{S}^\ell$ and $\mathbb{S}^\ell_>$ respectively designate the set of all matrices with q rows and ℓ columns, the set of all square matrices of order ℓ, the set of all symmetric matrices of order ℓ, and the set of all positive definite symmetric matrices of order ℓ. For vectors $\boldsymbol{v} = (v_i)$ and matrices $A = (A_{ij})$, we define the norms

$$\|\boldsymbol{v}\| = \sum_i |v_i| \quad \text{and} \quad \|A\| := \sum_{i,j} |A_{ij}|.$$

A generic point in $\mathbb{R}^2$ is denoted $x = (x_1, x_2)$ and partial derivatives of first and second order are denoted $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. An open ball with radius R centered at $x \in \mathbb{R}^2$ is denoted $B_R(x)$, or B_R if its center is irrelevant in the subsequent analysis.

The space of distributions over an open set $\Omega \subset \mathbb{R}^2$ is denoted $\mathcal{D}'(\Omega)$. The usual Sobolev spaces being denoted $W^{m,p}(\Omega)$, we let

$$W^{m,p}_{\mathrm{loc}}(\Omega) := \left\{ f \in \mathcal{D}'(\Omega);\ f \in W^{m,p}(U) \text{ for all open set } U \Subset \Omega \right\},$$

where the notation $U \Subset \Omega$ means that the closure of U in $\mathbb{R}^2$ is a compact subset of Ω. The closure in $W^{1,p}(\Omega)$ of the space of all indefinitely derivable functions with compact support in Ω is denoted $W^{1,p}_0(\Omega)$. If $p > 2$, the classes of functions in $W^{1,p}(\Omega)$ are identified with their continuous representatives, as in the Sobolev imbedding theorem (see, e.g., Adams [1]). For matrix-valued and vector-valued function spaces, we shall use the notations $W^{m,p}(\Omega; \mathbb{M}^{q\times\ell})$, $W^{m,p}(\Omega; \mathbb{R}^\ell)$, etc.

The Lebesgue spaces $L^p(\Omega; \mathbb{R}^d)$ and $L^p(\Omega; \mathbb{M}^{q\times\ell})$ are equipped with the norms

$$\|\boldsymbol{v}\|_{L^p(\Omega)} = \sum_i \|v_i\|_{L^p(\Omega)} \quad \text{and} \quad \|A\|_{L^p(\Omega)} = \sum_{i,j} \|A_{ij}\|_{L^p(\Omega)},$$

and the Sobolev spaces $W^{1,p}(\Omega; \mathbb{M}^{q\times\ell})$ and $W^{2,p}(\Omega; \mathbb{M}^{q\times\ell})$ are equipped with the norms

$$\|Y\|_{W^{1,p}(\Omega)} = \|Y\|_{L^p(\Omega)} + \sum_i \|\partial_i Y\|_{L^p(\Omega)} \quad \text{and} \quad \|Y\|_{W^{2,p}(\Omega)} = \|Y\|_{W^{1,p}(\Omega)} + \sum_{i,j} \|\partial_{ij} Y\|_{L^p(\Omega)}.$$

The following theorem gathers the Morrey and Sobolev inequalities with explicit constants that will be used in the next sections:

Theorem 1.1. *Let $B_R \subset \mathbb{R}^2$ be an open ball of radius $R > 0$ and let $p > 2$. Then there exists constants $C_1, C_2 > 0$ depending only on p such that*

$$\left|u(x)-u(y)\right| \leqslant C_1 R^{1-2/p}\|\nabla u\|_{L^p(B_R)} \quad \textit{for all } u \in W^{1,p}(B_R) \textit{ and all } x, y \in B_R \text{ (Morrey's inequality)},$$
$$\|u\|_{L^{2p/(p-2)}(B_R)} \leqslant C_2 R^{1-2/p}\|\nabla u\|_{L^2(B_R)} \quad \textit{for all } u \in W_0^{1,2}(B_R) \text{ (Sobolev inequality)}.$$

2. Stability of Pfaff systems

We establish here the following stability result for Pfaff systems with L^p_{loc}-coefficients defined over an open subset of $\mathbb{R}^2$. However, the same analysis can be carried out in higher dimensions without difficulty.

Theorem 2.1. *Let Ω be a connected open subset of $\mathbb{R}^2$, let $x^0 \in \Omega$, let $p > 2$, let $A_i^n \in L^p(\Omega; \mathbb{M}^\ell)$ and $Y^n \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q\times\ell})$ be sequences of matrix fields that satisfy the Pfaff systems*

$$\partial_i Y^n = Y^n A_i^n \quad \textit{in } \mathcal{D}'\big(\Omega; \mathbb{M}^{q\times\ell}\big), \quad n \in \mathbb{N},$$

and assume that there exists a constant M such that $\sum_i \|A_i^n\|_{L^p(\Omega)} + \|Y^n(x^0)\| \leqslant M$ for all $n \in \mathbb{N}$. Then, for each open set $K \Subset \Omega$, there exist a constant $C > 0$ such that

$$\big\|Y^n - Y^m\big\|_{W^{1,p}(K)} \leqslant C\left(\sum_i \big\|A_i^n - A_i^m\big\|_{L^p(\Omega)} + \big\|Y^n(x^0) - Y^m(x^0)\big\|\right) \quad \textit{for all } n, m \in \mathbb{N}.$$

Proof. Fix any open ball $B_R = B_R(x) \Subset \Omega$, where $x \in K$ and $2C_1 M R^{1-2/p} < 1$ (C_1 is the constant appearing in Theorem 1.1). Using Morrey's inequality (see Theorem 1.1), viz

$$\big\|Y^n - Y^m\big\|_{L^\infty(B_R)} \leqslant \big\|\big(Y^n - Y^m\big)(x)\big\| + C_1 R^{1-2/p} \sum_i \big\|\partial_i\big(Y^n - Y^m\big)\big\|_{L^p(B_R)}, \tag{2}$$

and the relation $\partial_i(Y^n - Y^m) = (Y^n - Y^m)A_i^n + Y^m(A_i^n - A_i^m)$, we obtain on the one hand that

$$\sum_i \big\|\partial_i\big(Y^n - Y^m\big)\big\|_{L^p(B_R)} \leqslant 2M\big\|\big(Y^n - Y^m\big)(x)\big\| + 2\big\|Y^m\big\|_{L^\infty(B_R)} \sum_i \big\|A_i^n - A_i^m\big\|_{L^p(B_R)}. \tag{3}$$

Using again Morrey's inequality together with relations $\partial_i Y^m = Y^m A_i^m$ and $2C_1 M R^{1-2/p} < 1$, we deduce on the other hand that

$$\big\|Y^m\big\|_{L^\infty(B_R)} \leqslant \big\|Y^m(x)\big\| + C_1 R^{1-2/p} \sum_i \big\|\partial_i Y^m\big\|_{L^p(B_R)} \leqslant 2\big\|Y^m(x)\big\|,$$

and, by joining the point x to x^0 by a broken line formed by N segments of length $< R$, that

$$\big\|Y^m\big\|_{L^\infty(B_R)} \leqslant 2^N \big\|Y^m(x^0)\big\| \leqslant 2^N M,$$

where the number N depends only on x_0, Ω and K. Using this inequality in inequality (3) gives

$$\sum_i \big\|\partial_i\big(Y^n - Y^m\big)\big\|_{L^p(B_R)} \leqslant 2M\big\|\big(Y^n - Y^m\big)(x)\big\| + 2^{N+1} M \sum_i \big\|A_i^n - A_i^m\big\|_{L^p(B_R)}. \tag{4}$$

Then we infer from inequalities (2) and (4) that there exists a constant $C > 0$ such that

$$\big\|Y^n - Y^m\big\|_{W^{1,p}(B_R(x))} \leqslant C\left(\big\|\big(Y^n - Y^m\big)(x)\big\| + \sum_i \big\|A_i^n - A_i^m\big\|_{L^p(\Omega)}\right).$$

Let now the open set $K \Subset \Omega$ be covered with a finite number of balls of radius R. By joining the center x of any such ball to the given point x^0 with a broken line formed by N segments of length $< R$ (the number N depends only on x_0, Ω and K), we show by a recursion argument that there exists another constant C independent of n, m such that

$$\|Y^n - Y^m\|_{W^{1,p}(B_R(x))} \leqslant C\Big(\big\|(Y^n - Y^m)(x^0)\big\| + \sum_i \|A_i^n - A_i^m\|_{L^p(\Omega)}\Big).$$

Since this inequality is valid for any ball $B_R(x)$ in the chosen covering of K, summing all such inequalities gives the announced inequality. □

An immediate consequence of Theorem 2.1 is the following uniqueness result:

Corollary 2.2. *Let Ω be an connected open subset of $\mathbb{R}^2$, let $p > 2$, and let there be given matrix fields $A_i \in L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$ and $Y, \widetilde{Y} \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q\times\ell})$ that satisfy the relations*

$$\partial_i Y = Y A_i \quad \textit{and} \quad \partial_i \widetilde{Y} = \widetilde{Y} A_i \quad \textit{in } \mathcal{D}'\big(\Omega; \mathbb{M}^{q\times\ell}\big).$$

Assume that there exists a point $x^0 \in \Omega$ such that $Y(x^0) = \widetilde{Y}(x^0)$. Then $Y(x) = \widetilde{Y}(x)$ for all $x \in \Omega$.

3. Existence of the solution to Pfaff systems with L^p coefficients in dimension two

Let $\Omega \subset \mathbb{R}^2$ be a connected and simply-connected open set and let there be given a point $x^0 \in \Omega$ and a matrix $Y^0 \in \mathbb{M}^{q\times\ell}$. Then it is well-known (see, e.g., Thomas [7]), that the (Cauchy problem associated with the) Pfaff system

$$\begin{aligned} &\partial_i Y = Y A_i \quad \text{in } \Omega,\ i \in \{1, 2\},\\ &Y(x^0) = Y^0, \end{aligned}$$

has a unique solution if the coefficients A_i belong to the space $\mathcal{C}^1(\Omega; \mathbb{M}^\ell)$ and satisfy the compatibility condition

$$\partial_1 A_2 + A_1 A_2 = \partial_2 A_1 + A_2 A_1 \quad \text{in } \Omega.$$

This result has been subsequently improved by Hartman and Wintner [3] (under the assumption that $A_i \in \mathcal{C}^0(\Omega; \mathbb{M}^\ell)$) and by Mardare [4] (under the assumption that $A_i \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$). Our objective is to establish an existence and uniqueness result under the assumption that $A_i \in L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$, $p > 2$. The key ingredient in establishing this result is the following lemma.

Lemma 3.1. *Let Ω be an open subset of $\mathbb{R}^2$, let $p > 2$, and let matrix fields $A_i \in L^p(\Omega; \mathbb{M}^\ell)$ be given that satisfy the relations*

$$\partial_1 A_2 + A_1 A_2 = \partial_2 A_1 + A_2 A_1 \quad \textit{in } \mathcal{D}'\big(\Omega; \mathbb{M}^\ell\big). \tag{5}$$

Then, for each open ball $B_R \subset \Omega$ whose radius satisfies

$$R < \min\big(1, \big\{C(p)\big(\|A_1\|_{L^p(\Omega)} + \|A_2\|_{L^p(\Omega)}\big)\big\}^{p/(2-p)}\big), \tag{6}$$

where $C(p)$ is a constant depending only on p, there exist sequences of matrix fields $A_i^n \in \mathcal{C}^\infty(\overline{B}_R; \mathbb{M}^\ell)$, $n \in \mathbb{N}$, that satisfy the relations

$$\begin{aligned} &\partial_1 A_2^n + A_1^n A_2^n = \partial_2 A_1^n + A_2^n A_1^n \quad \textit{in } B_R,\\ &A_i^n \to A_i \quad \textit{in } L^p\big(B_R; \mathbb{M}^\ell\big) \textit{ as } n \to \infty. \end{aligned}$$

Proof. The key of the proof is the following change of unknowns:

$$A_1 = \partial_1 U - \partial_2 V \quad \text{and} \quad A_2 = \partial_2 U + \partial_1 V \quad \text{in } B_R,$$

where $U \in W^{1,p}(B_R, \mathbb{M}^\ell)$ and $V \in W^{2,p/2}_{\gamma_0}(B_R, \mathbb{M}^\ell) := W^{2,p/2}(B_R, \mathbb{M}^\ell) \cap W^{1,p}_0(B_R, \mathbb{M}^\ell)$. This system has a solution that can be computed as follows: First, define $V \in W^{2,p/2}_{\gamma_0}(B_R, \mathbb{M}^\ell)$ as the solution to the Poisson equation

$$\Delta V = A_2 A_1 - A_1 A_2 \quad \text{in } \mathcal{D}'(B_R, \mathbb{M}^\ell),$$

then define $U \in W^{1,p}(B_R, \mathbb{M}^\ell)$ as a solution to the Poincaré system (assumption (5) is used here)

$$\partial_1 U = A_1 + \partial_2 V \quad \text{and} \quad \partial_2 U = A_2 - \partial_1 V.$$

Now, the approximating sequences for the fields A_i are defined in the following way: First, the field U is approximated with the smooth matrix fields $U^n \in \mathcal{C}^\infty(\overline{B}_R, \mathbb{M}^\ell)$ defined by taking the convolution of (an extension to $\mathbb{R}^2$ of) U with a sequence of mollifiers, so that

$$U^n \to U \quad \text{in } W^{1,p}\big(B_R, \mathbb{M}^\ell\big) \text{ as } n \to \infty.$$

Then the field V_n, $n \in \mathbb{N}$, is defined as the solution to the system

$$\begin{aligned}&\Delta V^n = \big(\partial_2 U^n + \partial_1 V^n\big)\big(\partial_1 U^n - \partial_2 V^n\big) - \big(\partial_1 U^n - \partial_2 V^n\big)\big(\partial_2 U^n + \partial_1 V^n\big) \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{M}^\ell\big),\\ &V^n = 0 \quad \text{on the boundary of } B_R.\end{aligned}$$

We prove that this nonlinear system has at least one solution of class $\mathcal{C}^\infty$ in $\overline{B}_R$ by using the implicit function theorem (see, e.g., Schwartz [6]) applied to the mapping

$$f : W^{1,p}\big(B_R; \mathbb{M}^\ell\big) \times W^{2,p/2}_{\gamma_0}\big(B_R, \mathbb{M}^\ell\big) \to L^{p/2}\big(B_R, \mathbb{M}^\ell\big),$$

defined by $f(X,Y) = \Delta Y - (\partial_2 X + \partial_1 Y)(\partial_1 X - \partial_2 Y) + (\partial_1 X - \partial_2 Y)(\partial_2 X + \partial_1 Y)$. To this end, we show that the mapping $\frac{\partial f}{\partial Y}(U,V)$ is an isomorphism from the space $W^{2,p/2}_{\gamma_0}(B_R, \mathbb{M}^\ell)$ to the space $L^{p/2}(B_R; \mathbb{M}^l)$ by using the Lax–Milgram lemma, Theorem 1.1, and assumption (6) on the size of the ball B_R. Consequently, there exist open subsets $O_1 \subset W^{1,p}(B_R; \mathbb{M}^\ell)$ and $O_2 \subset W^{2,p/2}_{\gamma_0}(B_R, \mathbb{M}^\ell)$ and a mapping $\varphi \in \mathcal{C}^1(O_1; O_2)$ such that $U \in O_1$, $V \in O_2$ and $\{(X,Y) \in O_1 \times O_2;\ f(X,Y) = 0\} = \{(X, \varphi(X));\ X \in O_1\}$. In particular, for $X = U^n$ there exists $V^n := \varphi(U^n)$ such that $f(U^n, V^n) = 0$. The regularity properties of second order elliptic partial differential equations (see, e.g., Gilbarg and Trudinger [2]) show that in fact $V^n \in \mathcal{C}^\infty(\overline{B}_R)$. Moreover, since φ is continuous and since $U^n \to U$ in $W^{1,p}(B_R; \mathbb{M}^\ell)$, it follows that $V^n \to V$ in $W^{2,p/2}(B_R; \mathbb{M}^\ell)$, hence in $W^{1,p}(B_R; \mathbb{M}^\ell)$ by the Sobolev imbedding theorem (see, e.g., Adams [1]).

Finally, we define the fields $A_1^n := \partial_1 U^n - \partial_2 V^n$ and $A_2^n := \partial_2 U^n + \partial_1 V^n$, and prove that they satisfy the required conditions of the lemma. □

We are now in a position to prove the main result of this Note.

Theorem 3.2. *Let Ω be a connected and simply connected open subset of $\mathbb{R}^2$, let $x^0 \in \Omega$, let $p > 2$, let $Y^0 \in \mathbb{M}^{q\times\ell}$, and let matrix fields $A_i \in L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$ be given that satisfy the relations*

$$\partial_1 A_2 + A_1 A_2 = \partial_2 A_1 + A_2 A_1 \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{M}^\ell\big).$$

Then the Pfaff system

$$\begin{aligned}&\partial_i Y = Y A_i \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{M}^{q\times\ell}\big),\\ &Y(x^0) = Y^0\end{aligned} \tag{7}$$

has one and only one solution $Y \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q\times\ell})$.

Proof. We first prove the following local existence result: For each open ball $B_r := B_r(x^0) \Subset \Omega$ whose radius satisfies relation (6) of the previous lemma, there exists a field $Y \in W^{1,p}(B_r; \mathbb{M}^{q\times\ell})$ that satisfies the Pfaff system

$$\begin{aligned} &\partial_i Y = Y A_i \quad \text{in } \mathcal{D}'\big(B_r; \mathbb{M}^{q\times\ell}\big),\\ &Y(x^0) = Y^0. \end{aligned} \tag{8}$$

We find this solution as the limit of a sequence of solutions to some Pfaff systems with smooth coefficients. For, fix an open ball $B_R \Subset \Omega$ such that $B_r \Subset B_R$. Then Lemma 3.1 shows that there exist sequences of matrix fields $A_1^n, A_2^n \in \mathcal{C}^\infty(\overline{B}_R; \mathbb{M}^\ell)$ that satisfy

$$\begin{aligned} &\partial_1 A_2^n + A_1^n A_2^n = \partial_2 A_1^n + A_2^n A_1^n \quad \text{in } B_R,\\ &A_1^n \to A_1 \quad \text{and} \quad A_2^n \to A_2 \quad \text{in } L^p\big(B_R; \mathbb{M}^\ell\big) \text{ as } n \to \infty. \end{aligned}$$

Since the coefficients A_1^n and A_2^n are smooth, the classical result on Pfaff systems (see, e.g., Thomas [7]) shows that there exists a matrix field $Y^n \in \mathcal{C}^\infty(\overline{B}_R; \mathbb{M}^{q\times\ell})$ that satisfies

$$\begin{aligned} &\partial_i Y^n = Y^n A_i^n \quad \text{in } B_R,\ i \in \{1,2\},\\ &Y^n(x^0) = Y^0. \end{aligned} \tag{9}$$

By the stability result of Theorem 2.1, there exists a constant $C > 0$ such that

$$\big\|Y^n - Y^m\big\|_{W^{1,p}(B_r)} \leqslant C \sum_i \big\|A_i^n - A_i^m\big\|_{L^p(B_R)} \quad \text{for all } m, n \in \mathbb{N},$$

which means that (Y^n) is a Cauchy sequence in the space $W^{1,p}(B_r; \mathbb{M}^{q\times\ell})$. Since this space is complete, there exists a field $Y \in W^{1,p}(B_r; \mathbb{M}^{q\times\ell})$ such that $Y^n \to Y$ in $W^{1,p}(B_r; \mathbb{M}^{q\times\ell})$ as $n \to \infty$. In addition, the Sobolev imbedding $W^{1,p}(B_r; \mathbb{M}^{q\times\ell}) \subset \mathcal{C}^0(B_r; \mathbb{M}^{q\times\ell})$ shows that $Y^n(x^0) \to Y(x^0)$ in $\mathbb{M}^{q\times\ell}$ as $n \to \infty$. Then we deduce that the field Y satisfies the Pfaff system (8) by passing to the limit as $n \to \infty$ in the equations of system (9).

Now, we define a global solution to the Pfaff system (7) as in the proof of Theorem 3.1 of [4], by glueing together some sequences of local solutions along curves starting from the given point x^0. We prove that this definition is unambiguous thanks to the uniqueness result of Corollary 2.2 and to the simply-connectedness of the set Ω.

That this solution is unique follows from Corollary 2.2. □

Remark 1. The assumption that $p > 2$ of the theorem is optimal since in order to properly define $Y(x^0)$, the space $W^{1,p}(\Omega)$ (to which the components of the matrix field Y belong) should be imbedded in the space of continous functions.

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