Dynamical Systems

An invertible contraction that is not C^1-linearizable

Hildebrando M. Rodrigues [a,1], J. Solà-Morales [b,2]

[a] *Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Caixa Postal 668, 13560-970, São Carlos, SP, Brazil*

[b] *Departament de Matemàtica Aplicada 1, Universitat Politècnica de Catalunya, Av. Diagonal 647, 08028 Barcelona, Spain*

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Abstract

We present an example of a smooth invertible contraction in an infinite-dimensional Hilbert space that is not locally C^1-linearizable near its fixed point. ***To cite this article: H.M. Rodrigues, J. Solà-Morales, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

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Résumé

Une contraction inversible qui n'est pas C^1-linéarisable. Nous présentons un exemple de contraction inversible et régulière dans un espace de Hilbert de dimension infinie qui n'est pas localement C^1-linéarisable autour de son point fixe. ***Pour citer cet article : H.M. Rodrigues, J. Solà-Morales, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

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1. Introduction and main result

An invertible contraction both in finite or infinite dimensions can always be linearized in the class C^0, by the well known Hartman–Grobman theorem, as it was proved in Pugh [7]. However, it seems that it was not known until now if this was also true for the linearization in the class C^1 (see Abbaci [2]) in the case of infinite dimensional Banach spaces. All the existing results on this case, that to our knowledge are those of Mora and Solà-Morales [6], Tan [11], and the three independent recent works ElBialy [3], Abbaci [1,2] and Rodrigues and Solà-Morales [9]

E-mail addresses: hmr@icmc.usp.br (H.M. Rodrigues), jc.sola-morales@upc.edu (J. Solà-Morales).

require extra hypotheses for an invertible contraction in infinite dimensions to be $\mathcal{C}^1$-linearizable. (We take this opportunity to apologize for having written our work [9] without knowledge of the existence of [3] and [1].) For example, in the paper [9] we proved the following theorem:

Theorem 1.1 (A linearization theorem for contractions). *Let X be a Banach space with the property that there exists a function ρ such that*

$$\rho \in \mathcal{C}^{1,1}(X, \mathbb{R}), \quad with \ \rho(z) = 1, \ when \ |z| \leqslant 1/2 \ and \ \rho(z) = 0, \ when \ |z| \geqslant 1. \tag{1}$$

Suppose that $L, L^{-1} \in \mathcal{L}(X)$. We assume that there exist real numbers v_i^-, v_i^+, $i = 1, \ldots, n$, such that $0 < v_n^- < v_n^+ < v_{n-1}^- < v_{n-1}^+ < \cdots < v_1^- < v_1^+ < 1$,

$$v_1^+ v_i^+ < v_i^-, \quad i = 1, \ldots, n \quad (nonresonance \ condition) \tag{2}$$

and $|\sigma(L)| \subset (v_n^-, v_n^+) \cup (v_{n-1}^-, v_{n-1}^+) \cup \cdots \cup (v_1^-, v_1^+)$.

Let $F = F(z)$ be a $\mathcal{C}^{1,1}$-function in a neighborhood of the origin with values in X, such that $F = 0$, $DF = 0$, at $z = 0$.

Then, for the map $T : z \mapsto z'$, $z' = Lz + F(z)$, there exists a $\mathcal{C}^1$-map $R : z \mapsto u$, $u = z + \psi(z)$, satisfying $\psi = 0$, $D\psi = 0$, at $z = 0$, such that $RTR^{-1} : u \mapsto u'$ has the form $u' = Lu$ in a sufficiently small neighborhood of the origin.

In the present Note we exhibit an example of a smooth invertible contraction that is not $\mathcal{C}^1$-linearizable. Our Banach space X will be the usual Hilbert space ℓ_2 of the square summable sequences, that obviously satisfies the condition (1). The linear operator L is an invertible contraction and the nonlinearity $F(z)$ will be a polynomial of degree 2. Also the set $|\sigma(L)|$ will consist of a single interval, and will not satisfy the nonresonance condition (2).

To our knowledge, this example appears to be the first one of this kind. It shows that the infinite-dimensional case is not like the finite-dimensional one, where all smooth invertible contractions can be linearized with a linearization of class $\mathcal{C}^1$ (see Hartman [5], or Chicone and Swanson [4]). Our example closes this question. As a consequence, to linearize a smooth invertible contraction in the class $\mathcal{C}^1$ in the infinite dimensional case, one can not avoid an extra hypothesis, perhaps like the nonresonance condition (2).

Our interest in the linearization problems started years ago, in the works Mora and Solà-Morales [6] and Rodrigues and Ruas Filho [8]. Recently, we have been working in $\mathcal{C}^1$-linearization in infinite dimensions, in the works Rodrigues and Solà-Morales [9], for the case of invertible contractions, and Rodrigues and Solà-Morales [10] where a case of a saddle point is studied. In both cases, applications to abstract wave equations have been presented. The present Note is a continuation of these previous works.

The main idea of our example appears in the following proposition. As the reader can appreciate, if one takes $\delta > |a^2 - a|$ and one makes the dimension n to grow unboundedly then the invariant manifold will grow without bound.

Proposition 1.2. *Let $0 < a < 1$ and $\varepsilon, \delta \in \mathbb{R}$ be positive numbers. Consider the map $(x, \xi_1, \ldots, \xi_n) \mapsto (x', \xi_1', \ldots, \xi_n')$ in $\mathbb{R}^{n+1}$ defined by:*

$$x' = ax, \quad \xi_k' = a\xi_k + \delta\xi_{k+1}, \quad for \ k = 1, 2, \ldots, n-1, \quad and \quad \xi_n' = a\xi_n + \varepsilon x^2. \tag{3}$$

If $\xi_i = \phi_i(x)$, $i = 1, \ldots, n$, defines a local invariant curve for the above map, differentiable at $x = 0$, such that $\phi_i(0) = 0$, $\phi_i'(0) = 0$, then $\phi_i(x) = \delta^{n-i}\varepsilon x^2/(a^2 - a)^{n-i+1}$, and in particular $|\phi_1(x)| \geqslant \delta^{n-1}\varepsilon x^2/|a^2 - a|^n$.

Let us introduce some notation. Let us write $\mathbf{y}_n := (y_{n,1}, \ldots, y_{n,n})$ for a generic vector of $\mathbb{R}^n$, and define the linear map $J_n : \mathbb{R}^n \to \mathbb{R}^n$ by $J_n\mathbf{y}_n := (y_{n,2}, \ldots, y_{n,n}, 0)$. Let I_n be the identity in $\mathbb{R}^n$. We will consider the linear map $L_n := aI_n + \delta J_n$ for some given scalars a, δ.

Let us also write $\mathbf{z} := (x, \mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_n, \ldots)$ for a generic vector of ℓ_2, and define $L : \ell_2 \to \ell_2$ by $L\mathbf{z} := (ax, L_2\mathbf{y}_2, L_3\mathbf{y}_3, \ldots, L_n\mathbf{y}_n, \ldots)$. Define also the quadratic maps $\mathbf{f}_n : \mathbb{R} \to \mathbb{R}^n$ by $\mathbf{f}_n(x) = (0, 0, \ldots, x^2)$ and finally $F : \ell_2 \to \ell_2$ by $F(\mathbf{z}) := (0, \varepsilon_2\mathbf{f}_2(x), \varepsilon_3\mathbf{f}_3(x), \ldots, \varepsilon_n\mathbf{f}_n(x), \ldots)$ for a given sequence (ε_n).

The following is our main result:

Theorem 1.3. *Let $\varepsilon > 0$ and $\varepsilon_n := \varepsilon/n$. Under the hypothesis,*

$$0 < a < 1, \quad a - a^2 < \delta < \min\{1 - a, a\} \tag{4}$$

the operator L is a contraction on ℓ_2, satisfies $|\sigma(L)| = [a - \delta, a + \delta]$, and the polynomial map of degree 2 defined in ℓ_2 by

$$z' = Lz + F(z),$$

is not $\mathcal{C}^1$-linearizable in any neighborhood of $z = 0$.

Remark 1. Observe that (4) implies that $(a + \delta)^2 > a - \delta$, so the nonresonance condition (2) is not satisfied.

Remark 2. To prove the Theorem we will call $T := L + F$ and we will suppose that a local invertible map R exists such that R and R^{-1} are of class $\mathcal{C}^1$ with $RTR^{-1} = L$, and then we will arrive to a contradiction. However, following carefully the proof one can see that to arrive to a contradiction we do not need even to require R and R^{-1} to be of class $\mathcal{C}^1$, but merely R and R^{-1} to be differentiable at $z = 0$.

2. Proof of Theorem 1.3

Lemma 2.1. *If $0 < a < 1$ and $r \in \mathbb{R}$, then the functional equation,*

$$\phi(ax) = a\phi(x) + rx^2 \tag{5}$$

has a unique local solution ϕ differentiable at $x = 0$, with $\phi(0) = 0$ and $\phi'(0) = 0$. This solution is given by $\phi(x) = rx^2/(a^2 - a)$.

Proof. Since $rx^2/(a^2 - a)$ is a particular solution, the other solutions would be of the form $\phi(x) = rx^2/(a^2 - a) + \phi_1(x)$, where ϕ_1 satisfies the homogeneous equation:

$$\phi_1(ax) = a\phi_1(x).$$

Let $x_0 \neq 0$. Then $\phi_1(a^n x_0) = a^n \phi_1(x_0)$. Since $\phi_1(0) = 0$ and $a^n x_0 \to 0$ as $n \to \infty$ we have

$$\phi_1'(0) = \lim_{n \to \infty} \frac{\phi_1(a^n x_0) - \phi_1(0)}{a^n x_0} = \frac{\phi_1(x_0)}{x_0}.$$

So $\phi_1(x_0) = 0$, since $\phi_1'(0) = 0$. $\square$

Proof of Proposition 1.2. We consider the functional equations satisfied by the ϕ_i. Starting with $i = n$ and using the previous lemma one obtains $\xi_n = \phi_n(x) = \varepsilon x^2/(a^2 - a)$. Substituting this expression in the $n - 1$ equation $\xi_n' = a\xi_{n-1} + \delta\xi_n$, using again the lemma one obtains:

$$\xi_{n-1} = \phi_{n-1}(x) = \frac{\varepsilon\delta}{(a^2 - a)^2}x^2.$$

Proceeding recursively one finally obtains: $\xi_1 = \phi_1(x) = \varepsilon\delta^{n-1}x^2/(a^2 - a)^n$. $\square$

Proof of the Theorem 1.3. Let us call $T := L + F$, and suppose that a local linearization map R exists such that $RTR^{-1} = L$. If both R and R^{-1} are differentiable at zero then from $RTR^{-1} = L$ one obtains that $DR(0)L = LDR(0)$ and so $DR(0)^{-1}RT(DR(0)^{-1}R)^{-1} = L$. So we can suppose that $DR(0) = I$.

Now, the linear subspace $\{(x, 0)\} \subset \ell_2$ is invariant by L, so $R^{-1}\{(x, 0)\}$ is invariant by T. This invariant curve can be expressed as $\{x, \Phi(x)\} \subset \ell_2$ in a neighborhood of zero, with $\Phi(0) = 0$, $D\Phi(0) = 0$.

Let $y = (\mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_n, \ldots)$, with $\mathbf{y}_n$ being as before a vector with n components, and let us also write $\Phi(x) = (\phi_2(x), \phi_3(x), \ldots)$. It is clear that $\mathbf{y}_n = \phi_n(x)$ will be an invariant manifold for the $n + 1$-dimensional system (3), with $\varepsilon = \varepsilon_n$. Then, because of the proposition, we have

$$\|\Phi(x)\| \geqslant \frac{\varepsilon_n \delta^{n-1}}{|a^2 - a|^n} x^2$$

and we obtain a contradiction by letting $n \to \infty$ if $\delta > |a^2 - a|$, as it was taken in (4).

Observe also that $\|L_n\| = \|aI_n + \delta J_n\| \leqslant a\|I_n\| + \delta\|J_n\| = a + \delta$.

Since this bound is independent of n we get that $\|L\| \leqslant a + \delta$, and because of (4), L is a contraction.

Next, one can show that the spectrum of L is the whole disk $|z - a| \leqslant \delta$ of the complex plane. We do not give all the details, but we merely say that this can be easily deduced from the following estimates:

Let $c \in \mathbb{C}$ and I_n, J_n defined as above. If $|c| > 1$, then $\|(cI_n + J_n)^{-1}\| \leqslant 1/(|c| - 1)$, and if $0 < |c| < 1$, then $\|(cI_n + J_n)^{-1}\| \geqslant 1/|c|^n$, for all $n \geqslant 2$. These estimates follow easily from the explicit formula

$$(cI_n + J_n)^{-1} = \frac{1}{c}I_n - \frac{1}{c^2}J_n + \frac{1}{c^3}J_n^2 + \cdots + \frac{(-1)^{n-1}}{c^n}J_n^{n-1}.$$

Remark 3. In order to obtain the slightly better result described in Remark 2, the previous proof requires a small modification: one has to prove that the set $R^{-1}\{(x, 0)\}$ is can be expressed as $\{x, \Phi(x)\}$. To do that one can write it as $\{(x + \phi_1(x)), \Phi(x)\}$ and then prove that $\phi_1 \equiv 0$, by using Lemma 2.1 with $r = 0$.

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