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Functional Analysis

Fixed point theorem in subsets of topological vector spaces

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Abstract

We prove new existence results of fixed points for upper semicontinuous multi-valued maps with not necessarily convex values. The definition domains are assumed to have the simplicial approximation property. ***To cite this article: Y. Askoura, C. Godet-Thobie, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Théorème de point fixe dans les sous ensembles d'espaces vectoriels topologiques. Nous montrons des nouveaux résultats d'existence de points fixes pour les applications multivoques à images non nécessairement convexes. Les ensembles de définition sont supposés avoir la propriété d'approximation simplicial. ***Pour citer cet article : Y. Askoura, C. Godet-Thobie, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction and preliminaries

In this Note, we are interested to establish the fixed point property for a relatively large class of multi-valued maps. Our aim is to relax the convexity condition imposed to the values of the maps and the local convexity imposed to their definition domains in several well known results. That is, we consider upper semicontinuous multi-valued maps and impose to their values to have contractible small neighborhoods. Our class contains maps with star-shaped values. On definition domains, we impose the so called *simplicial approximation property* [3].

Throughout this Note, spaces are assumed to be separated. If X is a topological space, 2^X denotes the set of all nonempty subsets of X. The abbreviations: t.v.s., u.s.c., s.a.p. respectively mean: topological vector space, upper semicontinuous, simplicial approximation property. The standard n-simplex Δ_N (where $N = \{0, \ldots, n\}$) is the

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convex hull of the canonical basis of $\mathbb{R}^{n+1}$ ($\mathbb{R}$ is the real line). If A is a subset of a given t.v.s., $co(A)$ denotes the convex hull of A and $\bar{A}$ the adherence of A. A topological space is said to be contractible if its identity map is homotopic to a constant map. By a finite dimensional subset of a t.v.s. E we mean a subset of E contained in a finite dimensional subspace of E.

The basic important notion used in this work is defined as follows:

Definition 1.1 [3]. A convex subset X of a t.v.s. E is said to have the *simplicial approximation property* (s.a.p.) provided: for each neighborhood V of the origin of E, there exists a finite dimensional compact convex subset K_V of X, such that for any simplex (a convex hull of finitely many vectors of X) P of X, there exists a continuous function $\rho : P \to K_V$, satisfying: $\rho(x) - x \in V$, for all $x \in P$.

The s.a.p. was introduced, in F-spaces (Fréchet spaces), by Kalton et al. [3], who have shown that it implies the fixed point property. About convex compact admissible sets, in the sense of Klee [4], it is obvious that they have the s.a.p. More generally, the weakly admissible sets in the sense of Nhu [5] have this property. Note that the definition of Nhu is given for metrizable t.v.s. The generalization of the weak admissibility to general t.v.s. and the proof that it enjoys the s.a.p. is given by Okon in [7], where he deduced from this fact the fixed point property for Kakutani maps (u.s.c. maps with compact convex values) in weak admissible subsets of t.v.s. The s.a.p. is also satisfied in Roberts spaces[1] [6].

Since, the convex compact subsets of locally convex topological vector spaces have the s.a.p. (they are admissible), the principal result of this work is a generalization of several well-known fixed point results concerning Kakutani maps (upper semicontinuous multi-valued maps with convex compact values), such as that of Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Hukuhara, Fan, Glicksberg, and others.

For reducing the convexity assumption in the values of the considered multi-valued maps, we use the following Horvath's result.

Theorem 1.2 ([2], Theorem 1). *Let E be a topological space, $N = \{0, \dots, n\}$, Δ_N the standard n-simplex and $F : 2^N \to 2^E$ a multi-valued map with nonempty contractible values such that,*

$$\forall J, J' \subset N,\ J \neq \emptyset,\ J \subset J' \implies F(J) \subset F(J').$$

Then, there exists a continuous function $f : \Delta_N \to E$ such that,

$$f(\Delta_J) \subset F(J), \quad \forall J \subset N.$$

2. Main results

We consider a t.v.s. E and we denote by $\mathcal{V}_0$ the set of all open balanced neighborhoods of the origin of E. Recall that a subset W of E is said to be balanced provided: $\lambda x \in W,\ \forall x \in W,\ \forall \lambda \in [-1, 1]$.

We use the following three lemmas to prove our first result.

Lemma 2.1. *Let X be a closed convex subset of E, P a compact finite dimensional subset of X and $T : P \to 2^X$ an u.s.c. multi-valued map such that,*

$$\exists Y \subseteq X,\ \exists Q \in \mathcal{V}_0,\ \forall Q' \subset Q,\ Q' \in \mathcal{V}_0,\ \forall x \in X,\ \left[T(x) + Q'\right] \cap Y \text{ is nonempty and contractible.}$$

Then, for every two elements U, V of $\mathcal{V}_0$, there exists a continuous function $f : P \to X$ such that

[1] Spaces constructed by the same method as the famous one constructed by Roberts [8], see [6] for a detailed definition.

$$\forall x \in P, \quad f(x) \in V + T\big[(x+U) \cap P\big].$$

In other words, f is a continuous selection of $x \mapsto V + T[(x+U) \cap P]$.

Lemma 2.2. *Let X be a closed convex subset of E, P a compact finite dimensional subset of X and $f : P \to X$ a continuous function. Then, for every V in $\mathcal{V}_0$, there exists a simplex D of X and a continuous function $\varphi : P \to D$ such that: $\varphi(x) \in V + f(x)$, for all $x \in P$.*

Lemma 2.3. *Let X be a compact subset of E, $T : X \to 2^X$ a closed (with closed graph) multi-valued map. If, for every V in $\mathcal{V}_0$, there exists a point x in X such that $(x + V) \cap T[(x + V) \cap X] \neq \emptyset$, then T has a fixed point.*

The following theorem is the principal result of this Note.

Theorem 2.4. *Let X be a compact convex subset of E possessing the s.a.p. and $T : X \to 2^X$ an u.s.c. multi-valued map with closed values such that:*

$$\exists Y \subseteq X,\ \exists Q \in \mathcal{V}_0,\ \forall Q' \subset Q,\ Q' \in \mathcal{V}_0,\ \forall x \in X, \big[T(x) + Q'\big] \cap Y \text{ is nonempty and contractible}. \tag{1}$$

Then, T has a fixed point.

Proof. According to Lemma 2.3, it suffices to prove that for any two elements $U, W \in \mathcal{V}_0$, there exists a point $x_0 \in X$ such that

$$(x_0 + W) \cap T\big[(x_0 + U) \cap X\big] \neq \emptyset.$$

Let $U, W \in \mathcal{V}_0$ and $V \in \mathcal{V}_0$ such that $V + V + V \subset W \cap Q$.

Since X has the simplicial approximation property, there exists a finite dimensional compact convex subset K_V of X such that, for every simplex P of X, there exists a continuous function $\rho : P \to K_V$ such that

$$\rho(x) - x \in V, \quad \forall x \in P. \tag{2}$$

From Lemma 2.1, there exists a continuous function $f : K_V \to X$ which is a selection of $x \mapsto T[(x + U) \cap K_V] + V$, i.e.

$$\forall x \in K_V, \quad f(x) \in T\big[(x+U) \cap K_V\big] + V. \tag{3}$$

The function f is a continuous single valued function, then Lemma 2.2 guarantees the existence of a simplex D of X and a continuous function $\varphi : K_V \to D$ which is a selection of $x \mapsto f(x) + V$, i.e.

$$\forall x \in K_V, \quad \varphi(x) \in f(x) + V. \tag{4}$$

Choose in (2), P identical to D. Then ρ is defined on D.

The function $\rho \circ \varphi$ ($K_V \xrightarrow{\varphi} D \xrightarrow{\rho} K_V$) possesses a fixed point, denote it by x_0.

We have $x_0 \in (\rho \circ \varphi)(x_0)$. Put $y_0 = \varphi(x_0)$. Then, $x_0 = \rho(y_0)$ which gives, according to (2),

$$x_0 - y_0 \in V. \tag{5}$$

Using (3) and (4), we obtain $y_0 \in [f(x_0) + V] \subset T[(x_0 + U) \cap K_V] + V + V$.

Then, $y_0 \in (T[(x_0 + U) \cap X] + V + V)$. Taking into account (5),

$$(x_0 + V) \cap \big(T\big[(x_0 + U) \cap X\big] + V + V\big) \neq \emptyset$$

or in another form,

$$(x_0 + V + V + V) \cap T\big[(x_0 + U) \cap X\big] \neq \emptyset.$$

Finally, $(x_0 + W) \cap T[(x_0 + U) \cap X] \neq \emptyset$. $\square$

Remark 1.

(i) The set Y in the condition (1) can be assumed to be identical to X. We simply introduce this set in order to make a weaker condition.

(ii) The condition (1) of Theorem 2.4 is satisfied if the values of T are star-shaped. In fact, It is easy to see that if A is a subset of E which is star-shaped from x_0 and $V \in \mathcal{V}_0$, then, $A+V$ is also star-shaped from x_0

$$\big(\forall t \in [0,1],\ \forall a \in A,\ \forall v \in V,\ \big[(1-t)x_0 + t(a+v)\big] = \big[(1-t)x_0 + ta + tv\big] \subset A + V\big).$$

Then, for all $x \in X$, for all $V \in \mathcal{V}_0$, $T(x)+V$ will be a star-shaped set from a given point of $T(x)$. $[T(x)+V] \cap X$ will be also star-shaped form this same point. Consequently, for all $x \in X$, for all $V \in \mathcal{V}_0$, $[T(x)+V] \cap X$ is contractible.

Theorem 2.5. *Suppose that E is metrizable. Let X be a convex compact subset of E possessing the s.a.p. Then, any u.s.c. multi-valued map $T : X \to 2^X$ with ∞-proximally connected values* ([1], *page* 15) *has a fixed point.*

Proof. The proof is the same as that of Theorem 2.4. Just, instead of Lemma 2.1 (where the condition (1) is used), we use the well known result concerning the approximation of multi-valued maps with ∞-proximally connected values ([1], Theorem 23.8, page 113). $\square$

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