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Numerical Analysis

A posteriori error estimation for the dual mixed finite element method of the Stokes problem

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Abstract

This Note presents an a posteriori error estimator of residual type for the stationary Stokes problem using the dual mixed FEM. We prove lower and upper error bounds with the explicit dependence of the viscosity parameter and without any regularity assumption on the solution. ***To cite this article: M. Farhloul et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Estimateur d'erreur a posteriori pour une méthode d'éléments finis mixte duale pour le problème de Stokes. Dans cette Note, nous présentons un estimateur a posteriori du type résidual pour le problème de Stokes stationnaire approché par une méthode d'éléments finis mixte duale. Nous établissons l'équivalence entre l'erreur et cet estimateur, la dépendance des constantes d'équivalence en fonction du paramètre de viscosité étant explicite. Cet équivalence est aussi établie sans aucune condition de régularité de la solution. ***Pour citer cet article : M. Farhloul et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Introduction

Les estimateurs d'erreur a posteriori pour des problèmes aux limites standards sont de nos jours bien compris et sont un outil indispensable pour leur approximation (voir par exemple [12]). L'analyse de tels estimateurs pour

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la méthode d'éléments finis mixtes pour des opérateurs elliptiques scalaires ou le système de l'élasticité a été faite dans [2,4,1,13,5] mais à notre connaissance une étude similaire pour la formulation mixte duale du problème de Stokes stationnaire n'a pas été effectuée. Le but de cette note est donc de faire cette analyse en proposant un estimateur d'erreur du type résiduel. Nous établissons ensuite l'équivalence entre l'erreur et cet estimateur, où la dépendance des constantes en fonction du paramètre de viscosité est explicite et sans aucune condition de régularité sur la solution.

Discrétisation du problème

Nous considèrons le problème de Stokes stationnaire (1) dans un domaine Ω borné à bord polygonal Γ. La formulation duale mixte de ce problème est maintenant bien connue [8,9] et consiste à trouver $((\sigma, p), u)$ dans $\Sigma \times M$ solution de (3), lorsque Σ et M sont définis par (2). Comme ce problème a une solution unique [8,9,11], son unique solution $((\sigma, p), u)$ est donnée par $\sigma = \nu\nabla u$, où (u, p) est l'unique solution de (1).

Le problème (3) est approché par le problème discret (4) lorsque $\Sigma_h \times M_h$ est un sous-espace de $\Sigma \times M$ construit à partir d'une triangulation régulière $\mathcal{T}$ du domaine et basé sur les les éléments finis de Raviart–Thomas de degré le plus bas [8,9,11].

Estimations d'erreur

Théorème 0.1. *Soient* $((\sigma, p), u) \in \Sigma \times M$ *la solution de* (3) *et* $((\sigma_h, p_h), u_h) \in \Sigma_h \times M_h$ *la solution de* (4)*. Alors il existe deux constantes strictement positives* C_1 *et* C_2 *indépendantes de* h *et* ν *telles que*

$$C_1\eta \leqslant \|\sigma - \sigma_h\| + \|p - p_h\| + \big\|\mathrm{div}\big(\sigma - \sigma_h - (p - p_h)\delta\big)\big\| + \nu\|u - u_h\| \leqslant C_2\eta,$$

où le résidu η *est défini par* (7) *et* $\|\cdot\|$ *désigne la norme de* $L^2(\Omega)$*.*

La preuve de la borne supérieure est basée sur une décomposition du type Helmholtz de $\sigma - \sigma_h$ [7] et les relations de type Galerkin (5) et (6). La démonstration de la borne inférieure est plus classique et est basée sur des intégrations par parties sur chaque élément et des inégalités inverses.

1. Introduction

Let us fix a bounded domain Ω of $\mathbb{R}^2$ with a polygonal boundary Γ. In this domain we consider the stationary Stokes system: Given a vector function $f = (f_1, f_2)$, find a vector function $u = (u_1, u_2)$ representing the velocity of the fluid and a scalar function p representing the pressure and satisfying

$$\begin{cases} -\nu\Delta u + \nabla p = f & \text{in } \Omega, \\ \mathrm{div}\, u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where $\nu > 0$ is the viscosity of the fluid. This problem has a unique (weak) solution $(u, p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ [10], where we recall that $L_0^2(\Omega) = \{q \in L^2(\Omega)\colon \int_\Omega q(x)\,\mathrm{d}x = 0\}$.

The dual mixed formulation of that problem is well known [8,9] (see also [11]). In order to recall it let us introduce the following notation and spaces: Introduce

$$M = \big[L^2(\Omega)\big]^2, \qquad \Sigma := \big\{(\tau, q) \in \big[L^2(\Omega)\big]^{2\times 2} \times L_0^2(\Omega)\colon \mathrm{div}(\tau - q\delta) \in M\big\}. \tag{2}$$

The space Σ is endowed with the natural norm $\|(\tau, q)\|_\Sigma^2 := \|\tau\|^2 + \|q\|^2 + \|\mathrm{div}(\tau - q\delta)\|^2$, where from now on the notation $\|\cdot\|$ (resp. $(\cdot,\cdot)$) means the $L^2(\Omega)$-norm ($L^2(\Omega)$-inner product) of matrix valued functions, vector valued functions or scalar functions according to the context.

With these notations we recall that the dual mixed formulation of problem (1) consists in finding $((\sigma, p), u)$ in $\Sigma \times M$ solution of

$$\begin{cases} \frac{1}{\nu}(\sigma, \tau) + \big(\operatorname{div}(\tau - q\delta), u\big) = 0, & \forall (\tau, q) \in \Sigma, \\ \big(\operatorname{div}(\sigma - p\delta), v\big) = -(f, v), & \forall v \in M, \end{cases} \tag{3}$$

where δ means the (2×2)-identity matrix. Since this problem has a unique solution [8,9,11], its unique solution $((\sigma, p), u)$ is given by $\sigma = \nu \nabla u$, where (u, p) is the unique solution of (1).

Problem (3) will be approximated in a conforming finite element space $\Sigma_h \times M_h$ of $\Sigma \times M$ based on a regular triangulation $\mathcal{T}$ of the domain and based on the Raviart–Thomas' elements of lowest degree [8,9,11]. The discrete problem has a unique discrete solution $((\sigma_h, p_h), u_h) \in \Sigma_h \times M_h$. We then consider an efficient and reliable a posteriori error estimator of residual type for the errors $\epsilon := \sigma - \sigma_h$, $r := p - p_h$ and $e := u - u_h$ in the natural norms $\|(\epsilon, r)\|_\Sigma$ and $\|e\|$.

A posteriori error estimators for standard elliptic boundary value problems is in our days well understood (see for instance [12] and the references cited there). The analysis of a posteriori error estimators for the mixed finite element method of a scalar second order elliptic equation or for the elasticity system were initiated in [2,4,1,13,5] but to our knowledge a similar analysis for the dual mixed formulation of the Stokes system was not yet done. Therefore the goal of this note is to make this analysis. We further take care about the dependence of the error bounds with respect to the viscosity parameter ν.

Let us point out that our analysis does not require any regularity assumption. This is not the case for the analysis for the elasticity problem in [5] which requires the H^2-regularity of the solution. Moreover, all the constants appearing in the error bounds are independent of the viscosity parameter ν.

Let us finish this introduction with some notations used below: For shortness the $L^2(D)$-norm will be denoted by $\|\cdot\|_D$. For a vector valued function $u = (u_1, u_2)$ the notations ∇u and $\operatorname{curl} u$ mean the gradient and the curl of u rowwise. On the other hand for a matrix valued function $\tau = (\tau_{ij})_{1 \leqslant i,j \leqslant 2}$, $\operatorname{div} \tau$ and $\operatorname{curl} \tau$ mean the divergence and the curl of τ row by row, namely $\operatorname{div} \tau = (\partial_1 \tau_{11} + \partial_2 \tau_{12}, \partial_1 \tau_{21} + \partial_2 \tau_{22})^\top$ and $\operatorname{curl} \tau = (\partial_1 \tau_{12} - \partial_2 \tau_{11}, \partial_1 \tau_{22} - \partial_2 \tau_{21})^\top$. Finally, the notation $a \lesssim b$ means the existence of a positive constant C (independent of $\mathcal{T}$, of the viscosity parameter ν and of the functions under consideration) such that $a \leqslant Cb$.

2. Discretization of the problem

The domain Ω is discretized by a conforming mesh $\mathcal{T}$, cf. [6] made of triangles and regular in Ciarlet's sense. Elements will be denoted by T and its edges are denoted by E.

For an edge E of an element T we fix one of the two normal vectors and denote it by $n_E = (n_x, n_y)^\top$. Introduce additionally the tangent vector $t_E := (-n_y, n_x)^\top$.

The jump of some function v across an edge E is then defined as

$$[\![v(y)]\!]_E := \lim_{\alpha \to +0} v(y + \alpha n_E) - v(y - \alpha n_E), \quad y \in E.$$

The approximation spaces are related to Raviart–Thomas' elements of lowest degree RT_0 [8,9,11], namely we approximate Σ and M respectively by

$$\Sigma_h := \big\{(\tau_h, q_h) \in \Sigma \colon\ q_{h|T} \in \mathbb{P}_0(T),\ \tau_{h|T} \in \big[RT_0(T)\big]^2,\ \forall T \in \mathcal{T}\big\},$$
$$M_h := \big\{v_h \in M \colon\ v_{h|T} \in \big[\mathbb{P}_0(T)\big]^2,\ \forall T \in \mathcal{T}\big\}.$$

The discrete problem associated with (3) is to find $((\sigma_h, p_h), u_h) \in \Sigma_h \times M_h$ such that

$$\begin{cases} \frac{1}{\nu}(\sigma_h, \tau_h) + \big(\operatorname{div}(\tau_h - q_h\delta), u_h\big) = 0, & \forall (\tau_h, q_h) \in \Sigma_h, \\ \big(\operatorname{div}(\sigma_h - p_h\delta), v_h\big) = -(f, v_h), & \forall v_h \in M_h. \end{cases} \tag{4}$$

We recall that this problem has a unique solution [8,9,11].

Since $\operatorname{div}(\sigma_h - p_h\delta)$ belongs to M_h, the second identity of (4) is equivalent to

$$\operatorname{div}(\sigma_h - p_h\delta) = -P_h^0 f,$$

where P_h^0 is the $[L^2(\Omega)]^2$-orthogonal projection on M_h or equivalently

$$(P_h^0 f)_{|T} = \frac{1}{|T|}\int_T f(x)\,\mathrm{d}x, \quad \forall f \in M, T \in \mathcal{T}.$$

Therefore using (3), we obtain the relations

$$\frac{1}{\nu}(\epsilon,\tau) + \bigl(\operatorname{div}(\tau - q\delta), e\bigr) = -\frac{1}{\nu}(\sigma_h,\tau) - \bigl(\operatorname{div}(\tau - q\delta), u_h\bigr), \quad \forall(\tau,q) \in \Sigma, \tag{5}$$

$$\bigl(\operatorname{div}(\epsilon, -r\delta), v\bigr) = -(f - P_h^0 f, v), \quad \forall v \in M. \tag{6}$$

Due to (4), the right-hand sides of (5) and (6) are equal to zero when test functions are in Σ_h and M_h respectively.

3. Error estimations

Definition 3.1. Let $((\sigma_h, p_h), u_h)$ be the solution of (4). Then for any $T \in \mathcal{T}$, the local residual error estimator is defined by

$$\eta_T^2 := \bigl\| f + \operatorname{div}(\sigma_h - p_h\delta)\bigr\|_T^2 + \|\operatorname{tr}\sigma_h\|_T^2 + h_T^2\|\sigma_h\|_T^2 + \sum_{E\subset\partial T} h_E \bigl\| [\![\sigma_h \cdot t_E]\!]_E \bigr\|_E^2.$$

The global residual error estimator is simply

$$\eta^2 := \sum_{T\in\mathcal{T}} \eta_T^2. \tag{7}$$

Theorem 3.2 (Upper error bound). *The error is bounded globally from above by*

$$\|\epsilon\| + \|r\| + \bigl\|\operatorname{div}(\epsilon - r\delta)\bigr\| + \nu\|e\| \lesssim \eta. \tag{8}$$

Proof. We start with the estimate on ϵ. By Lemma 3.2 of [7] there exist $z \in [H_0^1(\Omega)]^2$, $q \in L_0^2(\Omega)$ and $\psi \in [H^1(\Omega)]^2$ such that $\operatorname{div} z = 0$ and $\epsilon = \nabla z - q\delta + \operatorname{curl}\psi$, with the estimate

$$\|\nabla z\| + \|\nabla\psi\| + \|q\| \lesssim \|\epsilon\|. \tag{9}$$

The above decomposition allows to write (reminding that $\operatorname{div} z = \operatorname{tr}\sigma = 0$)

$$\|\epsilon\|^2 = (\epsilon - r\delta, \nabla z) + (\operatorname{tr}\sigma_h, q) + (\epsilon, \operatorname{curl}\psi).$$

Applying Green's formula in the first term of this right-hand side and using (6), we obtain $(\epsilon - r\delta, \nabla z) = (f - P_h^0 f, z)$. In the identity (5) taking as test function (τ_h, q_h) the pair $(\operatorname{curl}\psi_h, 0)$, where $\psi_h = \mathrm{I}_{\mathrm{Cl}}\psi$ is the so-called Clément interpolant of ψ, we get $(\epsilon, \operatorname{curl}\psi_h) = 0$. These properties yield

$$\|\epsilon\|^2 = (f - P_h^0 f, z) + (\operatorname{tr}\sigma_h, q) + \bigl(\epsilon, \operatorname{curl}(\psi - \psi_h)\bigr).$$

Using Green's formula in the last term of this right-hand side we finally obtain

$$\|\epsilon\|^2 = (f - P_h^0 f, z) + (\operatorname{tr}\sigma_h, q) + \sum_E \int_E [\![\sigma_h \cdot t_E]\!]_E \cdot (\psi - \psi_h).$$

Cauchy–Schwarz's inequality and an interpolation error estimate for the Clément interpolant lead to

$$\|\epsilon\|^2 \lesssim \|f - P_h^0 f\|\|z\| + \|\mathrm{tr}\,\sigma_h\|\|q\| + \left(\sum_E h_E \left\|[\![\sigma_h \cdot t_E]\!]_E\right\|_E^2\right)^{1/2} \|\nabla\psi\|.$$

Using (9) and the so-called Poincaré's inequality, we conclude

$$\|\epsilon\| \lesssim \eta. \tag{10}$$

We now continue with the estimation of the norm of r. Let $\xi \in [H(\mathrm{div};\Omega)]^2$ be such that (see, e.g., [3]) $\mathrm{div}\,\xi = f - P_h^0 f$ in Ω, with the estimate

$$\|\xi\| + \|\mathrm{div}\,\xi\| \lesssim \|f - P_h^0 f\|. \tag{11}$$

The above identity combined with (6) lead to

$$\big(\mathrm{div}(\epsilon + \xi - r\delta), v\big) = 0, \quad \forall v \in M.$$

As $\int_\Omega r(x)\,\mathrm{d}x = 0$ by the proof of Proposition 3.2 of [9] (or [8, p. 92]), we deduce that

$$\|r\| \lesssim \|\epsilon + \xi\|.$$

By the triangular inequality and the estimate (11) we arrive at

$$\|r\| \lesssim \|\epsilon\| + \|f - P_h^0 f\|. \tag{12}$$

We now pass to the estimation of e. Let $\tau \in [H^1(\Omega)]^{2\times 2}$ be such that $\mathrm{div}\,\tau = e$ in Ω, with the property

$$\|\tau\|_{1,\Omega} \lesssim \|e\|. \tag{13}$$

Then we may write

$$\|e\|^2 = (u - u_h, \mathrm{div}\,\tau) = -\nu^{-1}(\sigma,\tau) - \big(u_h, \mathrm{div}(RT_0\tau)\big),$$

this last identity following from Green's formula and a well-known property of the Raviart–Thomas' interpolant RT_0. Using the first identity of (4) with the test function $(RT_0\tau, 0)$ (which belongs to Σ_h) we get

$$\|e\|^2 = -\nu^{-1}(\sigma,\tau) + \nu^{-1}(\sigma_h, RT_0\tau) = -\nu^{-1}(\sigma - \sigma_h, \tau) + \nu^{-1}(\sigma_h, RT_0\tau - \tau).$$

Now Cauchy–Schwarz's inequality and a standard interpolation error estimate of the Raviart–Thomas' interpolant [3] allow to obtain

$$\|e\|^2 \lesssim \nu^{-1}\left(\|\epsilon\| + \left(\sum_T h_T^2 \|\sigma_h\|_T^2\right)^{1/2}\right)\|\tau\|_{1,\Omega}.$$

By the estimate (13) we conclude

$$\nu\|e\| \lesssim \|\epsilon\| + \left(\sum_T h_T^2 \|\sigma_h\|_T^2\right)^{1/2}. \qquad \square \tag{14}$$

Theorem 3.3 (Lower error bound). *For all elements T, the following local lower error bound holds*:

$$\eta_T \lesssim \sum_{T'\cap T\neq\emptyset} \|\epsilon\|_{T'} + \nu\|e\|_T + \left\|\mathrm{div}(\epsilon - r\delta)\right\|_T. \tag{15}$$

Proof. We proceed in a relatively standard way using some elementwise integrations by parts, some inverse inequalities and taking into account the dependence on ν. $\square$

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