Numerical Analysis

A finite volume scheme for anisotropic diffusion problems

Robert Eymard [a], Thierry Gallouët [b], Raphaèle Herbin [b]

[a] *Université de Marne-la-Vallée, Cité Descartes, 5, boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France*
[b] *Université de Provence, 39, rue Joliot-Curie, 13453 Marseille, France*

Abstract

A new finite volume for the discretization of anisotropic diffusion problems on general unstructured meshes in any space dimension is presented. The convergence of the approximate solution and its discrete gradient is proven. The efficiency of the scheme is illustrated by numerical results. ***To cite this article: R. Eymard et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***
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Résumé

Un schéma volumes finis pour les problèmes anisotropes sur des maillages non structurés. On présente ici un nouveau schéma volumes finis pour la discrétisation des équations de diffusion anisotropes sur des maillages non structurés, pour toute dimension d'espace. On prouve la convergence de la solution approchée, ainsi que celle d'un gradient approché. La pertinence du schéma est illustrée par des résultats numériques. ***Pour citer cet article : R. Eymard et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***
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1. Introduction

Anisotropic diffusion problems are encountered in various scientific fields: geosciences, biology, finance. Well known discretization methods are finite differences, finite volumes and finite elements. The mathematical study of convergence is quite well known in the case of isotropic diffusion for the three methods, and is the object of numerous works in the case of anisotropic diffusion; a thorough mathematical theory is known in the finite element theory. For various reasons, such as the ease of implementation, construction of simple Voronoï meshes and the coupling of equations of different kind, finite volumes are often chosen in actual applications. Our aim here is to

construct a finite volume scheme for anisotropic problems and prove its convergence towards the solution of the continuous problem. We thus study the following problem: find an approximation of the variational weak solution $\bar{u} \in H_0^1(\Omega)$ to the following equation:

$$-\text{div}(\Lambda \nabla \bar{u}) = f \quad \text{in } \Omega, \tag{1}$$

with homogeneous Dirichlet boundary conditions, where Ω is a polygonal open bounded connected subset of $\mathbb{R}^d$, $d \in \mathbb{N}^*$, and $f \in L^2(\Omega)$. The function Λ is measurable from Ω to the set of symmetric $d \times d$ real matrices, the eigenvalues of which are included in $[\alpha(x), \beta(x)]$ where $\alpha, \beta \in L^\infty(\Omega)$ are such that $0 < \alpha_0 \leqslant \alpha(x) \leqslant \beta(x)$ for a.e. $x \in \Omega$.

The track that we follow here consists in building an approximate gradient, using the usual cell values of the discrete unknowns. Note that such an approximate gradient was already considered in [3]. However, in this latter work, the approximate gradient was constructed using the so-called Raviart–Thomas shape functions. Unfortunately, these functions have simple expressions only in the case of triangles and rectangles. Here, we shall develop a new approximate gradient which does not use these shape functions, and which is easy to compute on any admissible finite volume mesh.

2. The finite volume scheme

Let $\mathcal{T} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be an admissible finite volume discretization of Ω in the sense of [2], Definition 9.1, p. 762. An example of two neighbouring control volumes K and L of $\mathcal{M}$ is depicted in Fig. 1. The notations are identical to that of [2].

The size of the discretization is defined by: $\text{size}(\mathcal{T}) = \sup\{\text{diam}(K), K \in \mathcal{M}\}$.

The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\text{int}}$ (resp. $\mathcal{E}_{\text{ext}}$). For any $\sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L$ (resp. $\mathcal{E}_{\text{ext}}, \sigma \in \mathcal{E}_K$). For all $K \in \mathcal{M}$, we denote by $\mathcal{N}_K$ the subset of $\mathcal{M}$ of the neighbouring control volumes, and we denote by $\mathcal{E}_{K,\text{ext}} = \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}$.

For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we define $D_{K,\sigma} = \{tx_K + (1-t)y, t \in (0, 1), \ y \in \sigma\}$. For all $\sigma \in \mathcal{E}$, let x_σ denote the center of gravity of σ. We shall measure the regularity of the mesh through the function $\text{regul}(\mathcal{T})$ defined by

$$\text{regul}(\mathcal{T}) = \inf\left\{\frac{d_{K,\sigma}}{d(x_\sigma, x_K)}, K \in \mathcal{M}, \ \sigma \in \mathcal{E}_K\right\}. \tag{2}$$

We denote by $H_\mathcal{T}(\Omega) \subset L^2(\Omega)$ the space of piecewise constant functions on the control volumes. For all $w \in H_\mathcal{T}(\Omega)$ and for all $K \in \mathcal{M}$, we denote by w_K the constant value of w in K. For any function $\varphi \in C_c^\infty(\Omega)$, we denote by $P_\mathcal{M}\varphi \in H_\mathcal{T}(\Omega)$ the piecewise function equal to $\varphi(x_K)$ on cell K. For $(v, w) \in (H_\mathcal{T}(\Omega))^2$, we let:

$$[v, w]_{\mathcal{T},\alpha} = \sum_{K|L \in \mathcal{E}_{\text{int}}} \frac{\text{meas}(K|L)}{d_{KL}} \frac{(v_L - v_K)(w_L - w_K)}{\text{meas}(D_{K,K|L} \cup D_{K,K|L})} \int_{D_{K,K|L} \cup D_{K,K|L}} \alpha(x)\,dx$$

$$+ \sum_{\sigma \in \mathcal{E}_{\text{ext}}} \frac{m_\sigma}{d_{K,\sigma}} \frac{v_K w_K}{\text{meas}(D_{K,\sigma})} \int_{D_{K,\sigma}} \alpha(x)\,dx.$$

We define a norm in $H_\mathcal{T}(\Omega)$ by $\|w\|_\mathcal{T} = ([w, w]_{\mathcal{T},1})^{1/2}$ (where 1 denotes the constant function equal to 1). We define the discrete gradient $\nabla_\mathcal{T} : H_\mathcal{T}(\Omega) \to H_\mathcal{T}(\Omega)^d$, by:

$$\text{meas}(K)\nabla_\mathcal{T} w_K = \sum_{L \in \mathcal{N}_K} \frac{\text{meas}(K|L)}{d_{KL}}(x_{K|L} - x_K)(w_L - w_K) - \sum_{\sigma \in \mathcal{E}_{K,\text{ext}}} \frac{\text{meas}(\sigma)}{d_{K,\sigma}}(x_\sigma - x_K)w_K,$$
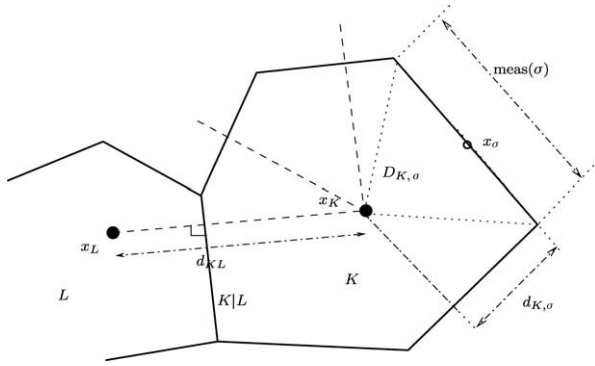
$$\forall K \in \mathcal{M}.$$
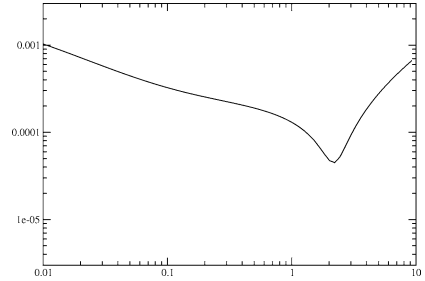
Fig. 1. Notations for an admissible mesh.

Fig. 2. L^2 norm of the error as a function of α.

We consider the finite volume scheme:

$$\begin{cases} u_{\mathcal{T}} \in H_{\mathcal{T}}(\Omega), \\ \displaystyle\int_{\Omega} \big(\Lambda(x) - \alpha(x)\,\mathrm{Id}\big)\nabla_{\mathcal{T}} u_{\mathcal{T}}(x) \cdot \nabla_{\mathcal{T}} v(x)\,\mathrm{d}x + [u_{\mathcal{T}}, v]_{\mathcal{T},\alpha} = \int_{\Omega} f(x)v(x)\,\mathrm{d}x, \quad \forall v \in H_{\mathcal{T}}(\Omega). \end{cases} \quad (3)$$

One may prove that there exists a unique solution $u_{\mathcal{T}}$ to (3), thanks to the following discrete H^1 estimate:

$$\alpha_0 \|u_{\mathcal{T}}\|_{\mathcal{T}} \leqslant \mathrm{diam}(\Omega) \|f\|_{L^2(\Omega)^d}. \quad (4)$$

3. Convergence results

Theorem 3.1 (Convergence of the finite volume scheme). *Let $\mathcal{T}$ be an admissible discretization of Ω. Let $u_{\mathcal{T}} \in H_{\mathcal{T}}(\Omega)$ be the solution to (3). Then $u_{\mathcal{T}}$ converges in $L^2(\Omega)$ to the weak solution $\bar{u}$ to the problem* (1)*, and the approximate gradient $\nabla_{\mathcal{T}} u_{\mathcal{T}}$ converges in $L^2(\Omega)^d$ to $\nabla\bar{u}$, as $\mathrm{size}(\mathcal{T}) \to 0$ provided that there exists $\rho > 0$ with $\mathrm{regul}(\mathcal{T}) \geqslant \rho$.*

Remark 1. Note that the present convergence result also gives the strong convergence of the present discrete gradient for the classical finite volume scheme in the isotropic case, which is given by (3) when $\Lambda = \alpha\,\mathrm{Id}$.

Sketch of proof of Theorem 3.1. From (4), we may extract from any sequence of approximate solutions a subsequence which converges to some $u \in H_0^1(\Omega)$. We shall show below that $u = \bar{u}$. Therefore u is unique. Hence we may consider any family of discretizations $\mathcal{T}$ with $\mathrm{regul}(\mathcal{T}) \geqslant \rho$, and $\mathrm{size}(\mathcal{T}) \to 0$.

Let $\varphi \in C_c^\infty(\Omega)$, set $v = P_{\mathcal{T}}\varphi$ in (3). The keypoints of the proof of convergence of $u_{\mathcal{T}}$ and $\nabla_{\mathcal{T}} u_{\mathcal{T}}$ are the following:

(i) $[u_{\mathcal{T}}, P_{\mathcal{T}}\varphi]_{\mathcal{T},\alpha} \to \int_{\Omega} \alpha(x)\nabla u(x) \cdot \nabla\varphi(x)\,\mathrm{d}x$,
(ii) $\nabla_{\mathcal{T}} u_{\mathcal{T}}$ tends to ∇u weakly in $L^2(\Omega)^d$,
(iii) $\nabla_{\mathcal{T}} P_{\mathcal{T}}\varphi$ tends to $\nabla\varphi$ in $L^2(\Omega)^d$.

Stage 1 is developed in [2,1]. Stage 2 may be shown by noting that, thanks to the orthogonality of $x_K x_L$ and $K|L$, we have

$$\sum_{K|L \in \mathcal{E}_{\mathrm{int}}} (u_L - u_K)\left[\mathbf{n}_{K,L} \int_{K|L} \varphi(x)d\gamma(x) - \frac{\mathrm{meas}(K|L)}{d_{KL}}\big((x_{K|L} - x_K)\varphi(x_K) + (x_L - x_{K|L})\varphi(x_L)\big)\right] \to 0,$$

which leads to

$$\int_\Omega P_\mathcal{T}\varphi(x)\nabla_\mathcal{T}u_\mathcal{T}(x)\,dx + \int_\Omega u_\mathcal{T}(x)\nabla\varphi(x)\,dx \to 0.$$

Stage 3 is a consequence of the following geometric property: for all $i, j = 1, \ldots, d$,

$$\int_K \operatorname{div}\!\big(x^{(i)}\mathbf{e}^{(j)}\big)\,dx = \delta_{ij}\operatorname{meas}(K) = \sum_{\sigma\in\mathcal{E}_K}\operatorname{meas}(\sigma)x_\sigma^{(i)}\mathbf{n}_{K,\sigma}\cdot\mathbf{e}^{(j)},$$

which follows thanks to the choice of x_σ as the center of gravity of σ. One then concludes to the convergence of $u_\mathcal{T}$ thanks to the fact that $\int_\Omega(\Lambda(x)-\alpha(x)\operatorname{Id})\nabla_\mathcal{T}u(x)\cdot\nabla_\mathcal{T}P_\mathcal{T}\varphi(x)\,dx$ tends to $\int_\Omega(\Lambda(x)-\alpha(x)\operatorname{Id})\nabla u(x)\cdot\nabla\varphi(x)\,dx$, and thus $u = \bar{u}$.

In order to prove the convergence of $\nabla_\mathcal{T}u_\mathcal{T}$, we consider a regular function ψ, and write the following inequality:

$$\|\nabla_\mathcal{T}u_\mathcal{T}-\nabla u\|_{L^2(\Omega)^d} \leqslant \|\nabla_\mathcal{T}u_\mathcal{T}-\nabla_\mathcal{T}P_\mathcal{T}\psi\|_{L^2(\Omega)^d} + \|\nabla_\mathcal{T}P_\mathcal{T}\psi-\nabla\psi\|_{L^2(\Omega)^d} + \|\nabla\psi-\nabla u\|_{L^2(\Omega)^d}.$$

Now, it may be shown that $[u_\mathcal{T}, u_\mathcal{T}]_{\mathcal{T},\alpha}$ tends to $\int_\Omega\alpha(x)\nabla u(x)^2\,dx$, as in [4], using weak convergence properties. Hence we get that

$$\limsup_{\operatorname{size}(\mathcal{T})\to 0}\|\nabla_\mathcal{T}u_\mathcal{T}-\nabla_\mathcal{T}P_\mathcal{T}\psi\|_{L^2(\Omega)^d} \leqslant C\|\nabla\psi-\nabla u\|_{L^2(\Omega)^d}.$$

Using stage 3, one has

$$\limsup_{\operatorname{size}(\mathcal{T})\to 0}\|\nabla_\mathcal{T}u_\mathcal{T}-\nabla u\|_{L^2(\Omega)^d} \leqslant (C+1)\|\nabla\psi-\nabla u\|_{L^2(\Omega)^d}.$$

Letting ψ tend to u in $H_0^1(\Omega)$, we conclude.

4. Numerical results

Fig. 2 shows the error between exact and approximate solution as a function of the parameter α for problem (1) with $\Omega = (0,1)\times(0,1)$, $\Lambda(x) = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$, and $\bar{u}(x) = x^{(1)}(1-x^{(1)})x^{(2)}(1-x^{(2)})$, using a uniform square 25×25 grid. We observe that a minimum is attained for a constant value of $\alpha(x)$ close to 2, the largest eigenvalue of $\Lambda(x)$. However, the above proof of convergence is only valid for $\alpha \leqslant 1$, the smallest eigenvalue of $\Lambda(x)$. Hence one should aim at proving the convergence for larger values of α. For this same problem, the order of convergence of the method with respect to the size of the mesh is close to 2 for the solution for both rectangles and triangles, and 1.5 (resp. 1) for its gradient in the case of rectangles (resp. triangles).

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