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C. R. Acad. Sci. Paris, Ser. I 339 (2004) 175–180

COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

A factorization method for elliptic problems in a circular domain

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Received 17 July 2003; accepted after revision 1 June 2004

Presented by Alain Bensoussan

Abstract

We present a method to factorize a second order elliptic boundary value problem in a circular domain, in a system of uncoupled first order initial value problems. We use a space invariant embedding technique along the radius of the circle, in a decreasing way. This technique is inspired in the temporal invariant embedding used by J.-L. Lions for the control of parabolic systems. The singularity at the origin for the initial value problems is studied. A formal calculation for more general star-shaped domains is presented. ***To cite this article: J. Henry et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Une méthode de factorisation de problèmes elliptiques dans un domaine circulaire. Nous présentons une méthode pour factoriser un problème aux limites elliptique du second ordre dans un domaine circulaire, en un système de problèmes découplés aux valeurs initiales du premier ordre. Nous utilisons une technique de plongement invariant en espace radiale, dans le sens décroissant. Cette technique s'inspire du plongement invariant temporel employé par J.-L. Lions pour le contrôle de systèmes paraboliques. La singularité à l'origine pour les problèmes aux valeurs initiales est étudiée. On prèsente aussi un calcul formel dans une situation plus générale de domaines étoilés. ***Pour citer cet article : J. Henry et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

On se propose d'appliquer spatialement la technique du plongement invariant de Bellman [2] à des problèmes aux limites elliptiques. Dans [3] la méthode est appliquée à l'équation de Poisson dans un cylindre. La famille de problèmes de la méthode de plongement invariant est définie par des sous-cylindres limités par une section variable et une condition aux limites additionnelle sur cette section. L'opérateur reliant les conditions aux limites

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.06.001

sur cette section (Dirichlet–Neumann ou Neumann–Dirichlet) vérifie une équation de Riccati en fonction d'une coordonnée le long de l'axe du cylindre. A l'aide de cet opérateur on peut transformer le problème aux limites en deux problèmes de type parabolique découplés dans des sens opposés. Cette factorisation peut être vue comme une généralisation en dimension infinie de la factorisation de Gauss LU par blocs de matrices bloc-triangulaires. On veut étudier ici le cas où la famille de surfaces qui limite les sous-domaines part du bord du domaine pour venir se réduire en un point. On considère donc le problème de Dirichlet pour l'équation de Poisson sur le disque Ω centré à l'origine et de rayon a, de bord Γ_a

$$(\mathcal{P}) \qquad -\Delta u = f, \quad \text{dans } \Omega; \quad u|_{\Gamma_a} = u_0. \tag{1}$$

Soit Ω_s le disque centré à l'origine et de rayon s, $0 < s < a$ de bord Γ_s. On suppose $u_0 \in H^{1/2}(\Gamma_a)$, $f \in L^2(\Omega)$, et une régularité supplémentaire sur un voisinage $\mathcal{O}$ de l'origine, $f \in \mathcal{C}^{0,\alpha}(\mathcal{O})$. Après passage en coordonnées polaires ($\hat{u}(\rho,\theta) = u(x_1,x_2)$ verifie (8)), on considère la famille de problèmes $(\widehat{\mathcal{P}}_{s,h})$ définis sur l'anneau $\Omega \setminus \Omega_s$ avec la condition aux limites additionnelle sur Γ_s $\frac{\partial \hat{u}_s}{\partial \rho}|_{\Gamma_s} = h$. Soient $\mathcal{I} =]0, 2\pi[$, et $H^1_{\rho,P}(\mathcal{I}) = \{v \mid v \in L^2(\mathcal{I}), \frac{1}{\rho}\frac{\partial v}{\partial \theta} \in L^2(\mathcal{I}), v(0) = v(2\pi)\}$. On choisit h dans $H^{1/2}_{\rho,P}(\mathcal{I})'$, où $H^{1/2}_{\rho,P}(\mathcal{I}) = [H^1_{\rho,P}(\mathcal{I}), L^2(\mathcal{I})]_{1/2}$. Par linéarité de (8) on a

$$\hat{u}_s|_{\Gamma_s} = P(s)h + r(s), \tag{2}$$

où $P(s)$ est l'opérateur Neumann–Dirichlet. Soit $X = \{\hat{v} \mid \hat{v} \in L^2_\rho(0,a; H^1_{\rho,P}(\mathcal{I})) \cap \frac{\partial \hat{u}}{\partial \rho} \in L^2_\rho(0,a; L^2(\mathcal{I}))\}$ où $L^2_\rho(0,a;Y)$ est l'espace des fonctions de ρ à valeurs dans Y de carré integrable dans $(0,a)$ avec le poids ρ. A partir de l'identité (2) écrite en suivant une solution de $(\widehat{\mathcal{P}}_{s_0,h})$, on obtient la factorisation du problème aux limites $(\widehat{\mathcal{P}})$ par

Théorème 0.1. *La solution $\hat{u}$ du problème $\widehat{\mathcal{P}}$ est la solution unique du système suivant de problèmes de Cauchy découplés, du premier ordre en ρ*

1. *pour tout $h, \bar{h}$ dans $L^2(\mathcal{I})$, l'opérateur autoadjoint $P(\rho) \in \mathcal{L}(L^2(\mathcal{I}), H^1_{\rho,P}(\mathcal{I})) \cap \mathcal{L}(H^{1/2}_{\rho,P}(\mathcal{I})', H^{1/2}_{\rho,P}(\mathcal{I})) \cap \mathcal{L}(H^1_{\rho,P}(\mathcal{I})', L^2(\mathcal{I}))$, $P \leqslant 0$, borné en ρ, vérifie l'équation de Riccati*

$$\left(\frac{\mathrm{d}P}{\mathrm{d}\rho}h, \bar{h}\right) + \left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta} \circ Ph, \frac{\partial}{\partial\theta} \circ P\bar{h}\right) - \left(\frac{1}{\rho}h, P\bar{h}\right) = (h, \bar{h}) \tag{3}$$

dans $\mathcal{D}'(0,a)$, avec la condition initiale $P(a) = 0$;

2. *pour tout h dans $L^2(\mathcal{I})$, $r \in X$ vérifie l'équation*

$$\left(\frac{\partial r}{\partial \rho}, h\right) + \left(\frac{1}{\rho^2}\frac{\partial r}{\partial \theta}, \frac{\partial}{\partial\theta} \circ Ph\right) = (\hat{f}, Ph) \tag{4}$$

dans $\mathcal{D}'(0,a)$, avec la condition initiale $r(a) = \hat{u}_0$;

3. *pour tout h dans $H^1_{\rho,P}(\mathcal{I})'$, $\hat{u} \in X$ vérifie l'équation*

$$-\left(\frac{\partial \hat{u}}{\partial \rho}, Ph\right) + \langle \hat{u}, h\rangle_{H^1_{\rho,P}(\mathcal{I}), H^1_{\rho,P}(\mathcal{I})'} = \langle r, h\rangle_{H^1_{\rho,P}(\mathcal{I}), H^1_{\rho,P}(\mathcal{I})'} \tag{5}$$

dans $\mathcal{D}'(0,a)$, avec la condition initiale $\hat{u}(0) = \lim_{\rho\to 0} r(\rho)$ dans $L^2(\mathcal{I})$ qui est constant en θ.

P et r sont ainsi définis de façon unique. Les Éqs. (3) *et* (4) *sont bien posées pour ρ décroissant de a à* 0 *et* (5) *pour ρ croissant de* 0 *à a.*

Pour étudier le comportement de u solution de (5) au voisinage de l'origine, on commence par étudier la factorisation du problème

$$(\mathcal{P}_\varepsilon) \qquad -\Delta u_\varepsilon = f, \quad \text{dans } \Omega \setminus \Omega_\varepsilon; \quad u_\varepsilon|_{\Gamma_a} = u_0; \quad \int_{\Gamma_\varepsilon} \frac{\partial u_\varepsilon}{\partial n}\,\mathrm{d}\Gamma = 0; \quad u_\varepsilon|_{\Gamma_\varepsilon} \text{ est constant} \tag{6}$$

où Ω_ε est de rayon ε, $0 < \varepsilon < a$ et l'on fait tendre ε vers 0. Afin de montrer que le résultat précédent n'est pas restreint aux domaines circulaires et aux coordonnées polaires on donne aussi le résultat d'un calcul similaire pour un domaine étoilé.

1. Introduction

The technique of invariant embedding was first introduced by Bellman [2] and was formally used by Angel and Bellman [1] in the resolution of Poisson's problem defined over a rectangle. J.L. Lions [4] gave a justification for this invariant embedding for the computation of the optimal feedback in the framework of Optimal Control of evolution equations of parabolic type. Henry and Ramos [3] presented a justification for the invariant embedding of Poisson's problem in a cylindrical domain. Embedding the problem in a family of similar ones defined on subcylinders bounded by a variable section, they obtained a factorization in two uncoupled problems of parabolic type, in opposite directions. The factorization uses a family of operators on functions of the section satisfying a Riccati equation. These operators relate the boundary conditions on the section (Dirichlet–Neumann or Neumann–Dirichlet). They showed that this can be seen as a generalization, up to infinite dimension, of the LU block factorization of matrices: solving the Riccati equation is analogous to computing the L and U factors for a block tridiagonal matrix and solving the two parabolic problems is related to solving the lower and upper triangular systems.

In this Note, we want to generalize this method to other types of geometries and, in particular, to the case where the family of surfaces which limits the sub-domains, starts on the outside boundary of the domain and shrinks to a point. We present here the simple situation where Ω (resp Ω_s) is a disk of $\mathbb{R}^2$ with radius a (resp. s) and centered on the origin and where the sub-domains defined by the invariant embedding are the annuli $\Omega \setminus \Omega_s$, $s \in (0, a)$.

2. Formulation of the problem and main result

We consider the Dirichlet problem for the Poisson equation defined over Ω.

$$(\mathcal{P}) \qquad -\Delta u = f, \quad \text{in } \Omega; \quad u|_{\Gamma_a} = u_0, \tag{7}$$

where Γ_s denotes the circle of radius s and center at the origin, and $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma_a)$. We assume the additional regularity around the origin $f \in \mathcal{C}^{0,\alpha}(\mathcal{O})$, $\mathcal{O}$ being a neighborhood of the origin. Introducing polar coordinates, $\hat{u}(\rho, \theta) = u(x_1, x_2)$ satisfies

$$(\widehat{\mathcal{P}}) \qquad -\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \hat{u}}{\partial \rho}\right) - \frac{1}{\rho^2}\frac{\partial^2 \hat{u}}{\partial \theta^2} = \hat{f}, \quad \text{in }]0, a[\times]0, 2\pi[; \quad \hat{u}|_{\Gamma_a} = \hat{u}_0, \quad \hat{u}(\rho, 0) = \hat{u}(\rho, 2\pi). \tag{8}$$

However, by doing this, we introduce a new difficulty on the problem since, with this new system of coordinates, we have a singularity at the origin. We embed problem $(\mathcal{P})$ in a family of similar problems $(\mathcal{P}_{s,h})$ defined on the annulus $\Omega \setminus \Omega_s$, $s \in]0, a[$ and satisfying an additional Neumann boundary condition in Γ_s; in terms of problem $(\widehat{\mathcal{P}})$, this can be written as $\frac{\partial \hat{u}_s}{\partial \rho}|_{\Gamma_s} = h$, where $\hat{u}_s$ is the solution of $\widehat{\mathcal{P}}_{s,h}$. Defining $\mathcal{I} =]0, 2\pi[$, $H^1_{\rho,P}(\mathcal{I})$ as the space of periodic functions v of θ, verifying $v \in L^2(\mathcal{I})$ and $\frac{1}{\rho}\frac{\partial v}{\partial \theta} \in L^2(\mathcal{I})$, we take $h \in H^{1/2}_{\rho,P}(\mathcal{I})'$, where $H^{1/2}_{\rho,P}(\mathcal{I}) = [H^1_{\rho,P}(\mathcal{I}), L^2(\mathcal{I})]_{1/2}$. By linearity of (8) we have

$$\hat{u}_s|_{\Gamma_s} = P(s)h + r(s), \tag{9}$$

where $P(s)$ is the Neumann to Dirichlet map. Let X be $X = \{\hat{v} \mid \hat{v} \in L^2_\rho(0, a; H^1_{\rho,P}(\mathcal{I})) \cap \frac{\partial \hat{u}}{\partial \rho} \in L^2_\rho(0, a; L^2(\mathcal{I}))\}$, where $L^2_\rho(0, a; Y)$ means functions of ρ with values in Y square integrable on $(0, a)$ with the weight ρ. Formally the factorization of problem (8) is obtained by taking $s = \rho > s_0$ and $h = \frac{\partial \hat{u}_{s_0}}{\partial \rho}$ in (9) for a solution $\hat{u}_{s_0}$ of $\widehat{\mathcal{P}}_{s_0,h}$. Then multiplying (9) by ρ, taking the derivative with respect to ρ, substituting the second derivative with respect to ρ from (8) and u from (9) yields

$$\left(\frac{\mathrm{d}P}{\mathrm{d}\rho} - P\frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}P - \frac{1}{\rho}P - I\right)\frac{\partial \hat{u}_{s_0}}{\partial \rho} + \frac{\partial r}{\partial \rho} - P\frac{1}{\rho^2}\frac{\partial^2 r}{\partial \theta^2} - P\hat{f} = 0, \tag{10}$$

for any h and so for any $\frac{\partial \hat{u}_{s_0}}{\partial \rho}$. From the boundary condition on Γ_a in (8) we obtain $P(a) = 0$ and $r(a) = \hat{u}_0$. The result is synthesized by the following theorem

Theorem 2.1. *The solution $\hat{u}$ of* (8) *is the unique solution of the following system of uncoupled, first order in ρ, initial value problems*

1. *for every $h, \bar{h}$ in $L^2(\mathcal{I})$, the selfadjoint operator $P(\rho) \in \mathcal{L}(L^2(\mathcal{I}), H^1_{\rho,P}(\mathcal{I})) \cap \mathcal{L}(H^{1/2}_{\rho,P}(\mathcal{I})', H^{1/2}_{\rho,P}(\mathcal{I})) \cap \mathcal{L}(H^1_{\rho,P}(\mathcal{I})', L^2(\mathcal{I}))$, $P \leqslant 0$, bounded as a function of ρ, satisfies the Riccati equation*

$$\left(\frac{\mathrm{d}P}{\mathrm{d}\rho}h, \bar{h}\right) + \left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta} \circ Ph, \frac{\partial}{\partial\theta} \circ P\bar{h}\right) - \left(\frac{1}{\rho}h, P\bar{h}\right) = (h, \bar{h}) \tag{11}$$

in $\mathcal{D}'(0,a)$, with the initial condition $P(a) = 0$;

2. *for every h in $L^2(\mathcal{I})$, $r \in X$ satisfies the equation*

$$\left(\frac{\partial r}{\partial\rho}, h\right) + \left(\frac{1}{\rho^2}\frac{\partial r}{\partial\theta}, \frac{\partial}{\partial\theta} \circ Ph\right) = (\hat{f}, Ph) \tag{12}$$

in $\mathcal{D}'(0,a)$, with the initial condition $r(a) = \hat{u}_0$;

3. *for every h in $H^1_{\rho,P}(\mathcal{I})'$, $\hat{u} \in X$ satisfies the equation*

$$-\left(\frac{\partial\hat{u}}{\partial\rho}, Ph\right) + \langle\hat{u}, h\rangle_{H^1_{\rho,P}(\mathcal{I}), H^1_{\rho,P}(\mathcal{I})'} = \langle r, h\rangle_{H^1_{\rho,P}(\mathcal{I}), H^1_{\rho,P}(\mathcal{I})'} \tag{13}$$

in $\mathcal{D}'(0,a)$, with the initial condition $\hat{u}(0) = \lim_{\rho\to 0} r(\rho)$ in $L^2(\mathcal{I})$ which is constant.

P and r thus defined are also unique. Eqs. (11) *and* (12) *are well posed for ρ decreasing from a to* 0 *and* (13) *is well posed for ρ increasing from* 0 *to a.*

This formulation of problem $(\mathcal{P})$ furnishes the Neumann to Dirichlet operator $P(s)$ for the annulus $\Omega \setminus \Omega_s$ which is of interest for various kinds of problems as domain decomposition or the definition of transparent boundary conditions. Furthermore Eq. (11) is intrinsic and only Eqs. (12) and (13) depend on the data f, u_0. For multiple resolution of $(\mathcal{P})$ with various data, only (12) and (13) need to be recomputed which is faster than solving the initial boundary value problem.

3. Factorization by invariant embedding (scheme of the proof)

In order to avoid the singularity created at the origin by polar coordinates we start by defining the following intermediate problem:

$$(\mathcal{P}_\varepsilon) \qquad -\Delta u_\varepsilon = f, \quad \text{in } \Omega \setminus \Omega_\varepsilon; \quad u_\varepsilon|_{\Gamma_a} = u_0; \quad \int_{\Gamma_\varepsilon} \frac{\partial u_\varepsilon}{\partial n}\,\mathrm{d}\Gamma = 0; \quad u_\varepsilon|_{\Gamma_\varepsilon} \text{ is constant} \tag{14}$$

where Ω_ε is a circular domain of radius $0 < \varepsilon < a$ and concentric with Ω. It's easy to see that this problem is well posed.

Proposition 3.1. *When $\varepsilon \to 0$, $\tilde{u}_\varepsilon$, defined as $\tilde{u}_\varepsilon = \begin{cases} u_\varepsilon, & \text{in } \Omega \setminus \Omega_\varepsilon, \\ u_\varepsilon = u_\varepsilon|_{\Gamma_\varepsilon}, & \text{in } \Omega_\varepsilon, \end{cases}$ where u_ε is the solution of problem* (14) *converges to u, solution of problem* (7), *in $H^1_0(\Omega)$.*

We can write problem (14) in polar coordinates restricting problem (8) over $]\varepsilon, a[\times\mathcal{I}$ and joining the boundary conditions $\hat{u}_\varepsilon|_{\Gamma_\varepsilon}$ constant, $\int_{\Gamma_\varepsilon} \frac{\partial\hat{u}_\varepsilon}{\partial\rho}\,\mathrm{d}\theta = 0$. Since $\frac{\partial\hat{u}_\varepsilon}{\partial\rho}|_{\Gamma_\varepsilon}$ is well determined through the conditions "$\hat{u}_\varepsilon|_{\Gamma_\varepsilon}$ constant" and "$\int_{\Gamma_\varepsilon} \frac{\partial\hat{u}_\varepsilon}{\partial\rho}\,\mathrm{d}\theta = 0$", it's clear that $(\widehat{\mathcal{P}}_\varepsilon)$ belongs to the family $(\widehat{\mathcal{P}}_{s,h})$ for $s = \varepsilon$. From (9), the solution $\hat{u}_\varepsilon$ of (14) in polar coordinates, satisfies the relation $\hat{u}_\varepsilon(\rho) = P(\rho)\frac{\partial\hat{u}_\varepsilon}{\partial\rho}|_{\Gamma_\rho} + r(\rho), \forall\rho \in [\varepsilon, a[$. Equations for P and r are derived as in (10). From these equations above, and respective initial conditions, we can obtain P and r. Knowing $P(\varepsilon)$ and $r(\varepsilon)$ we can determine uniquely $\hat{u}_\varepsilon(\varepsilon)$ satisfying "$\hat{u}_{\varepsilon|_{\Gamma_\varepsilon}}$ constant" and "$\int_{\Gamma_\varepsilon} \frac{\partial\hat{u}_\varepsilon}{\partial\rho}\,\mathrm{d}\theta = 0$". Let M and N be defined by $M = \{v \in (H^{1/2}_{\rho,P}(\mathcal{I}))' \mid \int_0^{2\pi} v\,\mathrm{d}\theta = 0\}$ and $N = M^{\perp} = \{v \in H^{1/2}_{\rho,P}(\mathcal{I}) \mid v \text{ is constant}\}$. One has $L^2(\mathcal{I}) = (M \cap L^2(\mathcal{I})) \oplus N$ and let Π_M and Π_N be the projection, for the $L^2(\mathcal{I})$ metrics, on each subspace respectively.

Proposition 3.2. *Given $r(\varepsilon) \in L^2(\mathcal{I})$, there exists a unique solution $\hat{u}_\varepsilon(\varepsilon) \in N$ and $\frac{\partial \hat{u}_\varepsilon}{\partial \rho}(\varepsilon) \in M$ for the equation $\hat{u}_\varepsilon(\varepsilon) = P(\varepsilon)\frac{\partial \hat{u}_\varepsilon}{\partial \rho}(\varepsilon) + r(\varepsilon)$. In particular, $\hat{u}_\varepsilon(\varepsilon) = \Pi_N r(\varepsilon)$.*

Using the Galerkin method as in [4,3] and adequate properties on the operator P and function r, we can justify the preceding formal calculation and we obtain, after passing to the limit when the dimension tends to infinity, the following result:

Proposition 3.3. *For every $h, \bar{h}$ in $L^2(\mathcal{I})$, the operator $P \in L^\infty(\varepsilon, a; \mathcal{L}(H^{1/2}_{\rho,P}(\mathcal{I})', H^{1/2}_{\rho,P}(\mathcal{I})))$ satisfies the following equation*

$$\left(\frac{\partial P}{\partial \rho} h, \bar{h}\right) + \left(\frac{1}{\rho^2}\frac{\partial}{\partial \theta} \circ Ph, \frac{\partial}{\partial \theta} \circ P\bar{h}\right) - \left(\frac{1}{\rho} h, P\bar{h}\right) = (h, \bar{h}), \tag{15}$$

in $\mathcal{D}'(\varepsilon, a)$, with $P(a) = 0$. The function r belongs to $X|_{\Omega\setminus\Omega_\varepsilon}$, satisfies $r(a) = \hat{u}_0$ and for every h in $L^2(\mathcal{I})$, satisfies in $\mathcal{D}'(\varepsilon, a)$ the following equation

$$\left(\frac{\partial r}{\partial \rho}, h\right) + \left(\frac{1}{\rho^2}\frac{\partial r}{\partial \theta}, \frac{\partial}{\partial \theta} \circ Ph\right) = (\hat{f}, Ph). \tag{16}$$

Since P and r do not depend on ε and thanks to the estimates on $P(\rho)$ and $r(\rho)$, we can take ε arbitrarily small and consequently consider the previous equalities defined on $\mathcal{D}'(0, a)$. Let $\|\cdot\|_\rho$ denote the norm in $\mathcal{L}(H^{1/2}_{\rho,P}(\mathcal{I})', H^{1/2}_{\rho,P}(\mathcal{I}))$. The following theorem gives the behaviour of P and r around the origin which provides the regularity claimed in Theorem 2.1:

Proposition 3.4. *For P satisfying* (15) *and $P(a) = 0$, we have $\lim_{\rho\to 0}\|P(\rho)\|_\rho = 1$. Furthermore $\lim_{\rho\to 0}\|P(\rho) - \rho(P_\infty\, o\, \Pi_M)\|_\rho = 0$, where P_∞ is the negative selfadjoint operator satisfying $-P_\infty \frac{\partial^2}{\partial\theta^2} P_\infty = I$.*

The solution r of (16) *and $r(a) = \hat{u}_0$, has a limit $r(0)$ constant with respect to θ: $\lim_{\rho\to 0}\|r(\rho) - r(0)\|_{L^2(\mathcal{I})} = 0$.*

It should be also remarked that, measured with the fixed norm $\|\cdot\|_{\mathcal{L}(L^2(\mathcal{I}),L^2(\mathcal{I}))}$, $P(\rho)$ goes to 0 as ρ goes to 0. Concerning the equation on $\hat{u}_\varepsilon$ we have

Proposition 3.5. *For every h in $H^1_{\rho,P}(\mathcal{I})'$, $\hat{u}_\varepsilon$ satisfies in $\mathcal{D}'(\varepsilon, a)$ the following equation*

$$-\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho}, Ph\right) + \langle \hat{u}_\varepsilon, h\rangle_{H^1_{\rho,P}(\mathcal{I}), H^1_{\rho,P}(\mathcal{I})'} = \langle r, h\rangle_{H^1_{\rho,P}(\mathcal{I}), H^1_{\rho,P}(\mathcal{I})'}, \tag{17}$$

with the initial condition $\hat{u}_\varepsilon(\varepsilon) = \Pi_N r(\varepsilon)$.

Using Proposition 3.1 we obtain the convergence in X of $\hat{u}_\varepsilon$ to $\hat{u}$ satisfying (13). Furthermore, thanks to the local regularity assumption on f near the origin which implies that $u \in \mathcal{C}^{2,\alpha}(\mathcal{O})$, one can prove that $\lim_{\varepsilon\to 0}\hat{u}_\varepsilon(\varepsilon) = \hat{u}(0)$ (the proof for the uniqueness of the solution of (13) uses the determination of $\hat{u}(0)$).

Remark 1. (a) Another factorization could be obtained by using an invariant embedding defined by the family of disks Ω_s. Here the main difficulty is to define the initial conditions for P and r at the origin.

(b) From the numerical viewpoint, a particular finite difference discretization of (11), (12) and (13) gives exactly the Gauss LU factorization of the matrix of the finite difference discretization of problem $(\widehat{\mathcal{P}})$. The same observation holds for finite element discretization. This is why this factorization can be viewed as an infinite dimensional extension of the Gauss factorization. Further, it may give clues to alternative discretization of (11), (12) and (13), resulting in new numerical schemes for $(\widehat{\mathcal{P}})$.

4. Star-shaped domains

We believe that this method is much more general than presented here and that it can be extended to higher dimensions, to other operators than the Laplacian and more general domains. For example we present now a formal calculation in the case of star-shaped domains. Let Ω be an open set containing the origin O, star-shaped with respect to O, with boundary $\Gamma = \partial\Omega$. We want to solve $(\mathcal{P})$ in Ω. Let α be the angle $(\vec{OM}, \vec{n})$ where M is a point on Γ and $\vec{n}$ is the outward normal to Ω at M. We assume that $-\pi/2 < \alpha_0 \leqslant \alpha \leqslant \alpha_1 < \pi/2$ and that the equation of Γ in polar coordinates is given by $\rho = \varphi(\theta)$. We consider the homothety of center O and ratio $0 < \tau < 1$, which transforms Ω to Ω_τ with boundary Γ_τ. We consider the following system of curvilinear coordinates: for $M \in \Omega$, (τ, t) are such that M', the image of M by a $1/\tau$ homothety, belongs to Γ and $t, 0 \leqslant t < t_0$, is the curvilinear abscissa of M' on Γ (t_0 is the length of Γ); $\hat{u}(\tau, t) = u(x_1, x_2)$. The family of problems $\mathcal{P}_{s,h}$ is now defined on $\Omega \setminus \Omega_s$ with the Neumann boundary condition $\frac{\partial \hat{u}}{\partial n}|_{\Gamma_s} = h$. By linearity, there exist $P(s)$ and $r(s)$ such that

$$\hat{u}|_{\Gamma_s} = P(s)h + r(s). \tag{18}$$

$\mathcal{I}$ now denotes $]0, t_0[$ and $(\cdot,\cdot)$ the scalar product in $L^2(\mathcal{I})$. Then P, r and $\hat{u}$ satisfy

1. for every $h, \bar{h}$ in $L^2(\mathcal{I})$, the selfadjoint operator $P(\tau) \in \mathcal{L}(L^2(\mathcal{I}), H^1_{\tau,P}(\mathcal{I})) \cap \mathcal{L}(H^{1/2}_{\tau,P}(\mathcal{I})', H^{1/2}_{\tau,P}(\mathcal{I})) \cap \mathcal{L}(H^1_{\tau,P}(\mathcal{I})', L^2(\mathcal{I}))$, $P \leqslant 0$, bounded as a function of τ, satisfies the Riccati equation

$$\begin{aligned}&\left(\frac{\mathrm{d}P}{\mathrm{d}\tau}h, \bar{h}\right) - \left(\frac{\hat{\varphi}\sin\alpha}{\tau}h, \frac{\partial}{\partial t}\circ P\bar{h}\right) - \left(\frac{\partial}{\partial t}\circ Ph, \frac{\hat{\varphi}\sin\alpha}{\tau}\bar{h}\right)\\&\quad + \left(\frac{\hat{\varphi}\cos\alpha}{\tau^2}\frac{\partial}{\partial t}\circ Ph, \frac{\partial}{\partial t}\circ P\bar{h}\right) - \left(\frac{1}{\tau}h, P\bar{h}\right) = (\hat{\varphi}\cos\alpha\, h, \bar{h})\end{aligned} \tag{19}$$

in $\mathcal{D}'(0,1)$, with the initial condition $P(1) = 0$;

2. for every h in $L^2(\mathcal{I})$, r satisfies the equation

$$\left(\frac{\partial r}{\partial \tau}, h\right) - \left(\frac{\hat{\varphi}\sin\alpha}{\tau}\frac{\partial r}{\partial t}, h\right) + \left(\frac{\hat{\varphi}\cos\alpha}{\tau^2}\frac{\partial r}{\partial t}, \frac{\partial}{\partial t}\circ Ph\right) = (\hat{\varphi}\cos\alpha\hat{f}, Ph) \tag{20}$$

in $\mathcal{D}'(0,1)$, with the initial condition $r(1) = \hat{u}_0$;

3. for every h in $H^1_{\tau,P}(\mathcal{I})'$, $\hat{u}$ satisfies the equation

$$-\left(\frac{1}{\hat{\varphi}\cos\alpha}\frac{\partial\hat{u}}{\partial\tau}, Ph\right) + \left(\frac{\tan\alpha}{\tau}\frac{\partial\hat{u}}{\partial t}, Ph\right) + \langle\hat{u}, h\rangle_{H^1_{\tau,P}(\mathcal{I}), H^1_{\tau,P}(\mathcal{I})'} = \langle r, h\rangle_{H^1_{\tau,P}(\mathcal{I}), H^1_{\tau,P}(\mathcal{I})'} \tag{21}$$

in $\mathcal{D}'(0,1)$, with the initial condition $\hat{u}(0) = \lim_{\tau\to 0} r(\tau)$ in $L^2(\mathcal{I})$ which is constant.

Acknowledgements

The third author would like to acknowledge the financial support of Fundação para a Ciência e a Tecnologia within the grant Praxis XXI, BD/ 21443/99. The second author was partially supported by Fundação para a Ciência e a Tecnologia (program POCTI/FEDER) via Centro de Matemática e Aplicações Fundamentais da Universidade de Lisboa.

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