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Probability Theory/Partial Differential Equations

Malliavin calculus for highly degenerate 2D stochastic Navier–Stokes equations

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Abstract

This Note mainly presents the results from “Malliavin calculus and the randomly forced Navier–Stokes equation” by J.C. Mattingly and E. Pardoux. It also contains a result from “Ergodicity of the degenerate stochastic 2D Navier–Stokes equation” by M. Hairer and J.C. Mattingly. We study the Navier–Stokes equation on the two-dimensional torus when forced by a finite dimensional Gaussian white noise. We give conditions under which the law of the solution at any time $t > 0$, projected on a finite dimensional subspace, has a smooth density with respect to Lebesgue measure. In particular, our results hold for specific choices of four dimensional Gaussian white noise. Under additional assumptions, we show that the preceding density is everywhere strictly positive. This Note’s results are a critical component in the ergodic results discussed in a future article. ***To cite this article: M. Hairer et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Calcul de Malliavin pour les équations de Navier–Stokes 2D stochastiques, hautement dégénérées. Cette Note présente essentiellement les résultats de l’article “Malliavin calculus and the randomly forced Navier–Stokes equation”, de J.C. Mattingly et E. Pardoux. Elle contient aussi un résultat de l’article “Ergodicity of the degenerate stochastic 2D Navier–Stokes equation”, de M. Hairer et J.C. Mattingly. Nous étudions l’équation de Navier–Stokes sur le tore bidimensionel, excitée par un bruit blanc gaussien de dimension finie. Nous donnons des conditions sous lesquelles la loi de la projection sur tout sous-espace de dimension finie de la solution à un instant $t > 0$ arbitraire a une densité régulière par rapport à la mesure de Lebesgue. Nos résultats sont en particulier vrais dans certains cas de bruit blanc gaussien de dimension quatre. Sous des hypothèses supplémentaires, nous montrons que la densité dont il est question ci-dessus est strictement positive partout. Les résultats de cette Note fournissent une part cruciale des arguments utilisés dans le second article cité ci-dessus, pour démontrer l’ergodicité de la solution. ***Pour citer cet article : M. Hairer et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

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1. Introduction

This note reports on recent progress made in [11,7] on the study of the two dimensional Navier–Stokes equation driven by an additive stochastic forcing. Recall that the Navier–Stokes equation describes the time evolution of an incompressible fluid. In vorticity form, it is given by

$$\begin{cases} \frac{\partial w}{\partial t}(t,x) + B(w,w)(t,x) = \nu\Delta w(t,x) + \frac{\partial W}{\partial t}(t,x), \\ w(0,x) = w_0(x), \end{cases} \tag{1}$$

where $x = (x_1, x_2) \in \mathbb{T}^2$, the two-dimensional torus $[0, 2\pi] \times [0, 2\pi]$, $\nu > 0$ is the viscosity constant, $\frac{\partial W}{\partial t}$ is a white-in-time stochastic forcing to be specified below, and $B(w, \widetilde{w})(x) = \sum_{i=1}^{2} (\mathcal{K}w)_i(x) \frac{\partial \widetilde{w}}{\partial x_i}(x)$, where $\mathcal{K}$ is the Biot–Savart integral operator which will be defined next. First, we define a convenient basis in which we will perform all explicit calculations. Setting $\mathbb{Z}^2_+ = \{(j_1, j_2) \in \mathbb{Z}^2\colon\ j_2 > 0\} \cup \{(j_1, j_2) \in \mathbb{Z}^2\colon\ j_1 > 0,\ j_2 = 0\}$, $\mathbb{Z}^2_- = -\mathbb{Z}^2_+$ and $\mathbb{Z}^2_0 = \mathbb{Z}^2_+ \cup \mathbb{Z}^2_-$, we define a real Fourier basis for functions on $\mathbb{T}^2$ with zero spatial mean by $e_k(x) = \sin(k \cdot x)$ if $k \in \mathbb{Z}^2_+$ and $\cos(k \cdot x)$ if $k \in \mathbb{Z}^2_-$. Write $w(t,x) = \sum_{k \in \mathbb{Z}^2_0} \alpha_k(t) e_k(x)$ for the expansion of the solution in this basis. With this notation, in the two-dimensional periodic setting, $\mathcal{K}(w) = \sum_{k \in \mathbb{Z}^2_0} \frac{k^\perp}{|k|^2} \alpha_k e_{-k}$, where $k^\perp = (-k_2, k_1)$. See for example [10] for more details on the deterministic vorticity formulation in a periodic domain. We use the vorticity formulation for simplicity, but all of our results can easily be translated into statements about the velocity formulation of the problem. We solve (1) on the space $\mathbb{L}^2 = \{f = \sum_{k \in \mathbb{Z}^2_0} a_k e_k\colon\ \sum |a_k|^2 < \infty\}$. For $f = \sum_{k \in \mathbb{Z}^2_0} a_k e_k$, we define the norms $\|f\|^2 = \sum |a_k|^2$ and $\|f\|_1^2 = \sum |k|^2 |a_k|^2$.

The emphasis of this note will be on forcing which directly excites only a few degrees of freedom. Such forcing is both of primary modeling interest and is technically the most difficult. Specifically we consider forcing of the form

$$W(t,x) = \sum_{k \in \mathcal{Z}_*} \sigma_k W_k(t) e_k(x). \tag{2}$$

Here $\mathcal{Z}_*$ is a finite subset of $\mathbb{Z}^2_0$, $\sigma_k > 0$, and $\{W_k\colon\ k \in \mathcal{Z}_*\}$ is a collection of mutually independent standard scalar Brownian Motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

This note mainly describes the results contained in Mattingly and Pardoux [11]. It gives conditions ensuring that any projection of the time t transition probability of the solution of (1) onto a finite dimensional subspace has a C^∞ density with respect to Lebesgue measure. The result is based on the Malliavin calculus. Under additional conditions, this density is shown to be everywhere positive. The techniques developed are quite general and we expect they can be applied to many nonlinear, stochastic partial differential equations with additive noise. These results provide a first step towards a truly infinite-dimensional version of Hörmander's celebrated 'sum of squares' theorem [9] in the setting of dissipative stochastic partial differential equations.

In a second paper, Hairer and Mattingly [7] give necessary and sufficient conditions for the main results and estimates of [11] to hold. These results are also described here. They then proceed to use these tools to build a theory which, when applied to (1), proves that it has a unique invariant measure under extremely general and essentially sharp assumptions. These results are described in the note [8]. To the best of the authors knowledge, that paper is the first to prove ergodicity of a nonlinear stochastic partial differential equation (SPDE) under assumptions comparable to those assumed when studying finite dimensional stochastic differential equations. The results in this note on Malliavin calculus and the spreading of the randomness are critical to proving the ergodic result.

2. The geometry of the forcing and cascade of randomness

The geometry of the forcing is encoded in the structure of $\mathcal{Z}_*$ from (2). As observed in [4], its structure gives information about how the randomness is spread throughout phase space by the nonlinearity. Define $\mathcal{Z}_0$ to be the symmetric, and hence translationally stationary part of the forcing set $\mathcal{Z}_*$, given by $\mathcal{Z}_0 = \mathcal{Z}_* \cap (-\mathcal{Z}_*)$. Then define the collection $\mathcal{Z}_n = \{\ell + j \in \mathbb{Z}_0^2\colon\ j \in \mathcal{Z}_0,\ \ell \in \mathcal{Z}_{n-1}$ with $\ell^\perp \cdot j \neq 0,\ |j| \neq |\ell|\}$ and lastly, $\mathcal{Z}_\infty = \bigcup_{n=1}^\infty \mathcal{Z}_n$.

$\mathcal{Z}_\infty$ captures the directions to which the randomness has spread. This can be understood in the following way. Denote by ∂_k the partial derivative in the direction e_k of the phase space and define (on a formal level) the first order differential operator $\mathcal{X} = \sum_{k \in \mathbb{Z}_0^2} (B(w,w)_k - \nu |k|^2)\partial_k$. Then the generator of the Markov process associated to (1) is formally given by $\mathcal{L} = \mathcal{X} + \frac{1}{2}\sum_{k \in \mathcal{Z}_*} \sigma_k \partial_k^2$. Note that $B(w,w)_k = \sum_{\ell,j} c_{k,j,\ell} w_\ell w_j$, where $c_{k,j,\ell} \neq 0$ if and only if $k \in \{j \pm \ell, -j \pm \ell\}$ and $\ell^\perp \cdot j \neq 0$, $|j| \neq |\ell|$. Therefore, all differential operators of the type ∂_k with $k \in \mathcal{Z}_\infty$ can be obtained as an iterated Lie bracket of finite length involving $\mathcal{X}$ and ∂_ℓ with $\ell \in \mathcal{Z}_*$. Since we want to ensure that all of the unstable directions are stochastically agitated, we seek conditions where $\mathcal{Z}_\infty = \mathbb{Z}_0^2$. Such conditions would then ensure that, on a formal level, the assumptions of Hörmander's theorem are met. The following essentially sharp characterization of this situation is given in [7].

Proposition 2.1. *One has $\mathcal{Z}_\infty = \mathbb{Z}_0^2$ if and only if both*:

(i) *Integer linear combinations of elements of $\mathcal{Z}_0$ generate $\mathbb{Z}_0^2$.*
(ii) *There exist at least two elements in $\mathcal{Z}_0$ with unequal Euclidean norm.*

This characterization is sharp in the sense that if $\mathcal{Z}_* = -\mathcal{Z}_*$ and one of the above two conditions fails, then there exists a non-trivial subspace of $\mathbb{L}^2$ which is left invariant under the dynamics of (1). Also notice that if $\mathcal{Z}_0 = \{(0,1),(0,-1),(1,1),(-1,-1)\}$ then Proposition 2.1 implies that $\mathcal{Z}_\infty = \mathbb{Z}_0^2$. Hence forcing four well chosen modes is sufficient to have the randomness move through the entire system. Of course one can also force a small number of modes centered elsewhere than at the origin and obtain the same effect. The next section and the second note discuss the implications of $\mathcal{Z}_\infty = \mathbb{Z}_0^2$.

3. Malliavin calculus and densities

We define $S_\infty = \mathrm{Span}(e_k\colon\ k \in \mathcal{Z}_\infty \cup \mathcal{Z}_*)$. One of the main results of [11] is the following:

Theorem 3.1. *For any $t > 0$ and any finite dimensional subspace S of S_∞, the law of the orthogonal projection $\Pi w(t,\cdot)$ of $w(t,\cdot)$ onto S is absolutely continuous with respect to the Lebesgue measure on S and has a C^∞ density.*

In [5], Eckmann and Hairer used Malliavin calculus to prove a version of Hörmander's 'sum of squares' theorem for a particular SPDE and deduce ergodicity. However, all of the techniques of that paper required that the forcing excite all but a finite number of directions and that the forcing be spatially rough as in [6,3]. The proof of Theorem 3.1 builds on ideas introduced into Malliavin calculus by Ocone in [12]. The central idea is an alternative representation of the Malliavin matrix of (1) using the time reversed adjoint of the linearization of (1). Ocone used this representation when the SPDE was linear in the initial data and the forcing. When the noise is additive, [11] extends that idea to the nonlinear case.

Let $J_{s,t}\xi$ be the solution of linearization of (1) at time t with initial condition ξ at time s, $s \leqslant t$. Let $\bar{J}^*_{s,t}\xi$ denote the solution to the $\mathbb{L}^2$-adjoint of the linearizion at time s, $s \leqslant t$, with terminal condition ξ at time t. Since the equation is time reversed, the adjoint is well posed. With this notation, the so-called 'Malliavin covariance matrix' $\mathcal{M}_t$ can be represented by

$$\langle \mathcal{M}_t \phi, \phi\rangle = \sum_{k \in \mathcal{Z}_*} \int_0^t \sigma_k^2 \langle J_{s,t} e_k, \phi\rangle^2 \, \mathrm{d}s = \sum_{k \in \mathcal{Z}_*} \int_0^t \sigma_k^2 \langle e_k, \bar{J}^*_{s,t}\phi\rangle^2 \, \mathrm{d}s \tag{3}$$

where $\phi \in \mathbb{L}^2$. The second of these representations is the one used in [11]. Because of the time reversal, the representation is not adapted to the filtration generated by W and new estimates concerning anticipating stochastic processes are required to obtain the needed estimates. Essentially one needs to show that the Malliavin matrix is non-degenerate on the subspace S and that the moments of the reciprocal of the norm of the Malliavin matrix on this subspace are finite. This is accomplished through the following estimate which also gives information about the separation of the randomness on large and small scales.

Theorem 3.2. *Let Π be the orthogonal projection of $\mathbb{L}^2$ onto a finite dimensional subspace of S_∞. For any $t > 0$, $\eta > 0$, $p \geqslant 1$, $M > 0$ and $K \in (0,1)$ there exist two constants $c = c(\nu, \eta, p, |\mathcal{Z}_*|, t, K, M, \Pi)$ and $\epsilon_0 = \epsilon_0(\nu, K, |\mathcal{Z}_*|, t, M, \Pi)$ such that for all $\epsilon \in (0, \epsilon_0]$,*

$$\mathbb{P}\Big(\inf_{\phi \in S(M,K,\Pi)} \langle \mathcal{M}_t \phi, \phi \rangle < \epsilon\Big) \leqslant c \exp\big(\eta \big\| w(0) \big\|^2\big) \epsilon^p$$

where $S(M, K, \Pi) = \{\phi \in S_\infty\colon\ \|\phi\| = 1,\ \|\phi\|_1 \leqslant M,\ \|\Pi\phi\| \geqslant K\}$.

With additional assumptions on the controllability of (1) conditions are also given ensuring the strict positivity of the density. This extends results of Ben Arous and Léandre [2] and Aida, Kusuoka and Stroock [1] to this setting. We refer the reader to [11] for the exact conditions and the details.

How these results can be used to prove ergodicity is described in [8].

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