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Dynamical Systems

On a theorem of Philip Hartman

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Abstract

We generalize Hartman's C^1 linearization theorem for local contractions and explain how to simplify its proof. ***To cite this article: B. Abbaci, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Sur un théorème de Philip Hartman. Nous généralisons le théorème de linéarisation C^1 des contractions locales dû à Hartman et expliquons comment en simplifier la démonstration. ***Pour citer cet article : B. Abbaci, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Hartman a montré [7] que tout germe $H:(\mathbf{R}^n,0)\to(\mathbf{R}^n,0)$ de difféomorphisme $C^{1+\mathrm{Lip}}$ ayant pour partie linéaire $DH(0)$ une contraction stricte est $C^{1+\text{Hölder}}$-linéarisable. Nous allons généraliser ce résultat à des germes éventuellement non inversibles en dimension éventuellement infinie.

Hypothèses et notations. Soient V une variété banachique telle que la topologie de l'espace de Banach modèle $E=T_pV$ peut être définie par une norme $C^{1+\mathrm{Lip}}$ en dehors de 0, et $h:(V,p)\to(V,p)$ un germe en $p\in V$ d'application $C^{1+\mathrm{Lip}}$. On suppose que le spectre de la différentielle $L:=T_ph$ de h en p admet une partition en un nombre fini de compacts $\sigma_0,\dots,\sigma_{n+1}$, non vides sauf éventuellement σ_{n+1}, tels que si l'on pose $a_i=\min\{|z|;\ z\in\sigma_i\}$, $b_i=\max\{|z|;\ z\in\sigma_i\}$ et $a_{n+1}=b_{n+1}=0$ si $\sigma_{n+1}=\emptyset$, on ait

$$0\leqslant a_{n+1}\leqslant b_{n+1}<a_n\cdots b_1<a_0\leqslant b_0<1\quad\text{et}\quad a_{i+1}\leqslant b_ib_0<a_i\quad(0\leqslant i\leqslant n).$$

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On note $E_0 \oplus \cdots \oplus E_n \oplus E_{n+1}$ la décomposition de E en somme directe de sous-espaces L-invariants tels que $\mathrm{Spec}(L|_{E_i}) = \sigma_i$, et $F_k := E_0 \oplus E_1 \oplus \cdots \oplus E_k$, $0 \leqslant k \leqslant n$.

Théorème 0.1. *Sous ces hypothèses, h est* $C^{1+\text{Hölder}}$-semi-linéarisable : *il existe un germe $\varphi_n : (V, p) \to (F_n, 0)$ d'application $C^{1+\text{Hölder}}$, tangent en p à la projection de E sur F_n, qui* semi-conjugue *h à $L|_{F_n}$: on a $\varphi_n \circ h = (L|_{F_n}) \circ \varphi_n$.*

En particulier, si $\sigma_{n+1} = \emptyset$, le germe h est inversible et $C^{1+\text{Hölder}}$-linéarisable puisque $\varphi := \varphi_n : (V, p) \to (E, 0)$ est alors tangent à l'identité et conjugue *h à L : on a $\varphi_* h := \varphi \circ h \circ \varphi^{-1} = L$.*

Remarque 1. La classe de Hölder de la dérivée de φ_n dépend [7,1,2] du spectre de L. L'hypothèse du théorème est toujours vérifiée lorsque la contraction stricte L est un opérateur compact ayant au moins une valeur propre non nulle. En effet, on groupe alors dans σ_0 les valeurs propres λ de L avec $|\lambda| > b_0^2 = \rho(L)^2$, puis on prend éventuellement pour b_1 le maximum des modules des valeurs propres non nulles restantes et l'on groupe dans σ_1 les valeurs propres λ de L avec $b_1 b_0 < |\lambda| \leqslant b_0^2$, etc. En dimension infinie, il faut choisir le n où s'arrête le processus ; en dimension finie, si L est inversible, on obtient le théorème de Hartman en s'arrêtant quand $\sigma_{n+1} = \emptyset$.

Idée de la démonstration [1,2]. Une carte locale tangente à l'identité permet de supposer que $(M, p) = (E, 0)$. On identifie E à $E_0 \times \cdots \times E_{n+1}$, on note $L_j := L|_{E_j} : E_j \to E_j$ et l'on prouve par récurrence sur k le résultat suivant, qui donne pour $k = n$ le Théorème 0.1 :

Lemme 0.2. *Pour $0 \leqslant k \leqslant n$, il existe un germe $\varphi_k : (E, 0) \to (F_k, 0)$ de transformation $C^{1+\text{Hölder}}$ tangent à la projection de E sur F_k et tel que le germe $\tilde{\varphi}_k : x \mapsto (\varphi_k(x), x_{k+1}, \ldots, x_{n+1})$ conjugue h à un germe $h_k : (E, 0) \to (E, 0)$ de la forme $h_k(x_0, \ldots, x_{n+1}) = (L_0 x_0, \ldots, L_k x_k, g_k(x_0, \ldots, x_{n+1}))$, où $\partial_{x_j} g_k$ est*

- *lipschitzienne pour $j > k$,*
- *hölderienne par rapport aux x_ℓ avec $\ell < k$ et lipschitzienne par rapport aux autres variables pour $j \leqslant k$.*

Posons $h_{-1} := h$ et $\tilde{\varphi}_{-1} := \mathrm{Id}_E$. Pour $-1 \leqslant k < n$, la construction de φ_{k+1} à partir de φ_k s'effectue en trouvant une semi-conjugaison π_{k+1} de h_k à L_{k+1}, le germe φ_{k+1} étant le composé de $\tilde{\varphi}_k$ et de $x \mapsto ((x_j)_{1 \leqslant j \leqslant k}, \pi_{k+1}(x))$. Pour cela, on remarque [3,4] que π_{k+1} est une semi-conjugaison de h_k à L_{k+1} si et seulement si son graphe est invariant par $h_k \times L_{k+1} : (x, y) \mapsto (h_k(x), L_{k+1}(y))$. Il s'agit donc de trouver un germe W_{k+1} de sous-variété (le graphe de π_{k+1}) invariant par $h_k \times L_{k+1}$ et tangent au graphe V_{k+1} de la projection $E \to E_{k+1}$.

En conjuguant $h_k \times L_{k+1}$ par l'automorphisme $\tau_k : (x, y) \mapsto (x, y - x_{k+1})$ de $E \times E_{k+1}$, on se ramène au cas où $V_{k+1} = E \times \{0\}$, et l'on applique à $I = F_k$ (avec la convention que $F_{-1} = \{0\}$), $J = E_{k+2} \times \cdots \times E_{n+1}$, $K = E_{k+1}$ et $\tilde{h} = \tau_k \circ (h_k \times L_{k+1}) \circ \tau_k^{-1}$ le résultat suivant :

Théorème 0.3. *Soit $\widetilde{E} = I \times J \times K$ un produit d'espaces de Banach tels que la norme de I soit $C^{1+\mathrm{Lip}}$ en dehors de 0. Soit $\tilde{h} : (\widetilde{E}, 0) \to (\widetilde{E}, 0)$ un germe C^1 de la forme $\tilde{h}(\theta, x, y) = (\Lambda\theta, f(\theta, x, y), g(\theta, x, y))$ vérifiant les conditions suivantes* :

(i) *$D\tilde{h}(0)$ est donnée par $D\tilde{h}(0)(\theta, x, y) = (\Lambda\theta, Bx + Dy, Cy)$, où Λ est un automorphisme de I, B un endomorphisme de J, D une application linéaire continue de K dans J et C un automorphisme de K tels qu'on ait*

$$\rho(C) = \rho(B) < \rho(\Lambda^{-1})^{-1} \leqslant \rho(\Lambda) < 1 \quad \textit{et} \quad \rho(C^{-1})\rho(B)\rho(\Lambda) < 1.$$

(ii) *Les dérivées partielles $\partial_x f$, $\partial_y f$, $\partial_x g$, $\partial_y g$ sont lipschitziennes.*

(iii) *Les dérivées partielles $\partial_\theta f$, $\partial_\theta g$ sont lipschitziennes par rapport à (x, y) et α-höldériennes par rapport à θ, avec $\alpha \in (0, 1)$.*

Il existe alors un germe en 0 *de variété* $\tilde{h}$*-invariante* $\widetilde{V}$ *de classe* $C^{1+\text{Hölder}}$ *tangente en* 0 *à* $I \times J \times \{0\}$*, qui est donc le graphe d'un germe d'application* $\Phi_0 : (I \times J, 0) \to (K, 0)$*. Ce germe est de classe* C^1*, tangent à* 0*, sa dérivée partielle* $\partial_x \Phi_0$ *est lipschitzienne par rapport à* (θ, x) *et sa dérivée partielle* $\partial_\theta \Phi_0$ *est lipschitzienne par rapport à* x *et höldérienne par rapport à* θ*.*

Pour $I = \{0\}$, ce théorème est un cas particulier d'un résultat de Chaperon [3,4]. On en déduit que le lemme pour $k = 0$, donc le théorème 0.1 pour $n = 0$, est vrai sans hypothèse supplémentaire sur E et sans perte de dérivabilité.

1. Introduction

According to a theorem by Hartman [7], every $C^{1+\text{Lip}}$ diffeomorphism germ $H : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ tangent at 0 to a strict contraction is $C^{1+\text{Hölder}}$-linearizable. We generalize his result to possibly non-invertible germs in possibly infinite dimension.

1.1. Hypotheses and notation

Let V be a Banach manifold such that the topology of its model Banach space $E = T_p V$ can be defined by a norm which is $C^{1+\text{Lip}}$ off 0, and let $h : (V, p) \to (V, p)$ be a $C^{1+\text{Lip}}$ map germ at $p \in V$. Assume that the spectrum of the differential $L := T_p h$ of h at p admits a partition into finitely many compact subsets $\sigma_0, \ldots, \sigma_{n+1}$ with the following property: setting $a_i := \min\{|z|;\ z \in \sigma_i\}$, $b_i := \max\{|z|;\ z \in \sigma_i\}$ and $a_i = b_i = 0$ if $\sigma_i = \emptyset$, we have

$$0 \leqslant a_{n+1} \leqslant b_{n+1} < a_n \cdots b_1 < a_0 \leqslant b_0 < 1 \quad \text{and} \quad a_{i+1} \leqslant b_i b_0 < a_i \ (0 \leqslant i \leqslant n)$$

hence in particular $\sigma_i \neq \emptyset$ for $i \leqslant n$. Denote by $E_0 \oplus \cdots \oplus E_n \oplus E_{n+1}$ the decomposition of E as the direct sum of L-invariant subspaces such that $\operatorname{Spec}(L|_{E_i}) = \sigma_i$, and let $F_k := E_0 \oplus E_1 \oplus \cdots \oplus E_k$, $0 \leqslant k \leqslant n$.

Theorem 1.1. *Under those hypotheses,* h *is* $C^{1+\text{Hölder}}$*-semi-linearizable: there exists a* $C^{1+\text{Hölder}}$ *map germ* $\varphi_n : (V, p) \to (F_n, 0)$*, tangent at* p *to the projection of* E *onto* F_n*, which* semi-conjugates h *to* $L|_{F_n}$*: we have* $\varphi_n \circ h = (L|_{F_n}) \circ \varphi_n$.

In particular, if $\sigma_{n+1} = \emptyset$*, the germ* h *is invertible and* $C^{1+\text{Hölder}}$-linearizable *since* $\varphi := \varphi_n : (V, p) \to (E, 0)$ *is tangent to the identity map and* conjugates h *to* L*: we have* $\varphi_* h := \varphi \circ h \circ \varphi^{-1} = L$.

Remark 1. The Hölder exponent of the derivative of φ_n depends on the spectrum of L [7,1,2]. The hypothesis of the theorem is satisfied when the strict contraction L is a compact operator having at least one non-zero eigenvalue.[1] Indeed, we can then take as σ_0 the set of those eigenvalues λ of L satisfying $|\lambda| > b_0^2 = \rho(L)^2$, as b_1 the maximum modulus of the remaining non-zero eigenvalues (if any) and then define σ_1 to be the set of those eigenvalues λ of L which satisfy $b_1 b_0 < |\lambda| \leqslant b_0^2$, etc. In the infinite dimensional case, one has to choose the n at which the process stops; in finite dimensions, when L is invertible, we get Hartman's theorem if we stop when $\sigma_{n+1} = \emptyset$.

2. On the proof of Theorem 1.1

See [1,2]. Using a local chart tangent to the identity map, we may assume that $(M, p) = (E, 0)$. We identify E to $E_0 \times \cdots \times E_{n+1}$, set $L_j := L|_{E_j} : E_j \to E_j$ and prove by induction on k the following result, which yields Theorem 1.1 for $k = n$:

[1] Hence applications to nonlinear parabolic partial differential equations [2].

Lemma 2.1. *For $0 \leqslant k \leqslant n$, there exists a $C^{1+\text{Hölder}}$ germ $\varphi_k : (E, 0) \to (F_k, 0)$, tangent to the projection $E \to F_k$, such that $\tilde{\varphi}_k : x \mapsto (\varphi_k(x), x_{k+1}, \ldots, x_{n+1})$ conjugates h to a germ $h_k : (E, 0) \to (E, 0)$ of the form $h_k(x_0, \ldots, x_{n+1}) = (L_0 x_0, \ldots, L_k x_k, g_k(x_0, \ldots, x_{n+1}))$, where $\partial_{x_j} g_k$ is*

- *Lipschitzian for $j > k$,*
- *Hölderian with respect to the x_ℓ's with $\ell < k$ and Lipschitzian with respect to the other variables for $j \leqslant k$.*

Set $h_{-1} := h$ and $\tilde{\varphi}_{-1} := \mathrm{Id}_E$. For $-1 \leqslant k < n$, we construct φ_{k+1} from φ_k in the following way: we find a semi-conjugacy π_{k+1} of h_k to L_{k+1} and define the germ φ_{k+1} to be the composed map of $\tilde{\varphi}_k$ and $x \mapsto ((x_j)_{1 \leqslant j \leqslant k}, \pi_{k+1}(x))$. Now, note [3,4] that π_{k+1} is a semi-conjugacy of h_k to L_{k+1} if and only if its graph is invariant by $h_k \times L_{k+1} : (x, y) \mapsto (h_k(x), L_{k+1}(y))$. Thus, we have to find a submanifold germ W_{k+1} (the graph of π_{k+1}) invariant by $h_k \times L_{k+1}$ and tangent to the graph V_{k+1} of the projection $E \to E_{k+1}$.

Conjugating $h_k \times L_{k+1}$ by the automorphism $\tau_k : (x, y) \mapsto (x, y - x_{k+1})$ of $E \times E_{k+1}$, we are in the situation where $V_{k+1} = E \times \{0\}$ and we can deduce Lemma 2.1 from the following result, applied to $I = F_k$ (setting $F_{-1} := \{0\}$), $J = E_{k+2} \times \cdots \times E_{n+1}$, $K = E_{k+1}$ and $\tilde{h} = \tau_k \circ (h_k \times L_{k+1}) \circ \tau_k^{-1}$:

Theorem 2.2. *Let $\widetilde{E} = I \times J \times K$ be a product of Banach spaces such that the norm of I is $C^{1+\text{Lip}}$ off 0, and let $\tilde{h} : (\widetilde{E}, 0) \to (\widetilde{E}, 0)$ be a C^1 map germ of the form $\tilde{h}(\theta, x, y) = (\Lambda\theta, f(\theta, x, y), g(\theta, x, y))$ with the following properties*:

(i) *$D\tilde{h}(0)$ is given by $D\tilde{h}(0)(\theta, x, y) = (\Lambda\theta, Bx + Dy, Cy)$, where Λ is an automorphism of I, B is an endomorphism of J, D is a continuous linear map of K into J and C is an automorphism of K, satisfying*

$$\rho(C) = \rho(B) < \rho(\Lambda^{-1})^{-1} \leqslant \rho(\Lambda) < 1 \quad \text{and} \quad \rho(C^{-1})\rho(B)\rho(\Lambda) < 1.$$

(ii) *The partial derivatives $\partial_x f$, $\partial_y f$, $\partial_x g$, $\partial_y g$ are Lipschitzian.*

(iii) *The partial derivatives $\partial_\theta f$, $\partial_\theta g$ are Lipschitzian with respect to (x, y) and Hölderian of exponent $\alpha \in (0, 1)$ with respect to θ.*

Then, there exists a $\tilde{h}$-invariant germ $\widetilde{V}$ at $0 \in \widetilde{E}$ of a $C^{1+\text{Hölder}}$ submanifold, tangent to $I \times J \times \{0\}$ and, therefore, graph of a $C^{1+\text{Hölder}}$ map germ $\Phi_0 : (I \times J, 0) \to (K, 0)$ tangent to 0, with the following properties: *the partial derivative $\partial_x \Phi_0$ is Lipschitzian with respect to (θ, x) and the partial derivative $\partial_\theta \Phi_0$ is Lipschitzian with respect to x and Hölderian with respect to θ.*

For $I = \{0\}$, Theorem 2.2 is a particular case of a result by Chaperon [3,4]. It follows that our lemma for $k = 0$ and therefore Theorem 1.1 for $n = 0$ is true for an arbitrary Banach space E with no loss of smoothness.

3. On the proof of Theorem 2.2

By the pseudo-unstable manifold theorem [6,5],[2] for every $\beta \in (0, \alpha]$ satisfying $\rho(\Lambda^{-1})^{-(1+\beta)} > \rho(B)$, there exists a $\tilde{h}$-invariant $C^{1+\beta}$ submanifold germ at $0 \in \widetilde{E}$ tangent to I. This submanifold germ is the graph of a $C^{1+\beta}$ germ $\chi : (I, 0) \to (J \times K, 0)$ tangent to 0. Hence, using the local change of variables $(\theta, x, y) \mapsto (\theta, (x, y) - \chi(\theta))$ we may add to the hypotheses of the theorem that $f(\theta, 0, 0) = 0$ and $g(\theta, 0, 0) = 0$.

[2] This is the place where we need the hypothesis on the norm of I.

Let a and b be positive numbers satisfying $b < \rho(C^{-1})^{-1} \leqslant \rho(B) < a$ and $a\,\rho(\Lambda) < b$. Choose the norms on I, J, K so that we have

$$b < |C^{-1}|^{-1} \leqslant |B| < a \quad \text{and} \quad a\,|\Lambda| < b.$$

Let $\mathcal{E}_{a,b}$ be the Banach space of those sequences $\underline{z} := (z_n)_{n\in\mathbf{N}} = (x_n, y_n)_{n\in\mathbf{N}}$ in $J \times K$ satisfying $|\underline{z}| := \sup\{\max\{a^{-n}|x_n|, b^{-n}|y_n|\};\ n \in \mathbf{N}\} < \infty$, endowed with the norm so defined. Identifying $\underline{z}$ to the pair $(\underline{x}, \underline{y})$, where $\underline{x} := (x_n)_{n\in\mathbf{N}}$ and $\underline{y} := (y_n)_{n\in\mathbf{N}}$, we may view $\mathcal{E}_{a,b}$ as the product of the two Banach spaces $\mathcal{J}_a$ and $\mathcal{K}_b$ consisting respectively of those sequences $\underline{x}$ in J with $|\underline{x}| := \sup\{a^{-n}|x_n|;\ n \in \mathbf{N}\} < \infty$ and those sequences $\underline{y}$ in K satisfying $|\underline{y}| := \sup\{b^{-n}|y_n|;\ n \in \mathbf{N}\} < \infty$. Set

$$F(\theta, \underline{z}) := \big(x_{n+1} - f(\Lambda^n\theta, x_n, y_n),\ y_{n+1} - g(\Lambda^n\theta, x_n, y_n)\big)_{n\in\mathbf{N}}. \tag{1}$$

It follows from our hypotheses that the $\tilde{h}$-invariant manifold germ $\tilde{V}$ we are looking for must be contained in the $\tilde{h}$-invariant germ of those (θ, z_0) such that z_0 is the first component of a sequence $\underline{z} \in \mathcal{E}_{a,b}$ with $F(\theta, \underline{z}) = \underline{0}$. Hence, Theorem 2.2 will be proven if we can show that the latter $\tilde{h}$-invariant germ is the graph of a map Φ_0 with the required properties.

Lemma 3.1. *Formula* (1) *defines a C^1 map germ $F : (I \times \mathcal{E}_{a,b}, (0, \underline{0})) \to (\mathcal{E}_{a,b}, \underline{0})$ with the following properties*

(i) *The partial derivative $\partial_{\underline{z}} F$ is Lipschitzian with respect to $(\theta, \underline{z})$.*
(ii) *The partial derivative $\partial_\theta F$ is Lipschitzian with respect to $\underline{z}$ and Hölderian with respect to θ.*
(iii) *The partial derivative of F at $(0, \underline{0})$ with respect to $(y_0, (z_n)_{n>0})$ is an isomorphism of the subspace $\mathcal{E}'_{a,b} := \{\underline{z} \in \mathcal{E}_{a,b} : x_0 = 0\}$ onto $\mathcal{E}_{a,b}$.*

It follows from Lemma 3.1 and the implicit function theorem that $F^{-1}(\underline{0})$, near $(0, \underline{0}) \in I \times \mathcal{E}_{a,b}$, is the graph of a function $\Phi : (I \times J, (0, 0)) \to (\mathcal{E}'_{a,b}, \underline{0})$. That function is of class C^1, its partial derivative $\partial_x \Phi$ is Lipschitzian with respect to (θ, x), its partial derivative $\partial_\theta \Phi$ is Lipschitzian with respect to x and Hölderian with respect to θ. Hence, the $\tilde{h}$-invariant manifold germ $\tilde{V}$ is the graph of the first component $\Phi_0 : (I \times J, 0) \to (K, 0)$ of Φ, and it has the required smoothness properties. We shall also show that it is tangent to 0, which completes the proof of Theorem 2.2.

Proof of Lemma 3.1(iii). Composing F with the projections, we see that (if it is differentiable) its differential is obtained by componentwise differentiation, hence

$$DF(0, \underline{0})\big(\theta, (x_n, y_n)_{n\in\mathbf{N}}\big) = (x_{n+1} - Bx_n - Dy_n,\ y_{n+1} - Cy_n)_{n\in\mathbf{N}}.$$

Setting $\mathcal{J}'_a := \{\underline{x} \in \mathcal{J}_a\colon x_0 = 0\}$, we have to show that $\mathcal{B} : \underline{x} \mapsto (x_{n+1} - Bx_n)_{n\in\mathbf{N}}$ is an isomorphism of $\mathcal{J}'_a$ onto $\mathcal{J}_a$, that $\mathcal{D} : \underline{y} \mapsto (Dy_n)_{n\in\mathbf{N}}$ is a continuous linear map and that $\mathcal{C} : \underline{y} \mapsto (y_{n+1} - Cy_n)_{n\in\mathbf{N}}$ is an automorphism of $\mathcal{K}_b$.

– $\mathcal{B}$ is obtained by composing the isomorphism $\mathcal{J}'_a \ni \underline{x} \mapsto (x_{n+1})_{n\in\mathbf{N}} \in \mathcal{J}_a$ and the endomorphism $\underline{x} \mapsto (x_0, (x_n - Bx_{n-1})_{n>0})$ of $\mathcal{J}_a$, which is an automorphism of the form "Id + strict contraction" since we have

$$a^{-n}|Bx_{n-1}| \leqslant a^{-1}|B|\big(a^{-(n-1)}|x_{n-1}|\big) \leqslant a^{-1}|B||\underline{x}| \quad \text{and} \quad a^{-1}|B| < 1.$$

– $\mathcal{D}$ is continuous since we have $a^{-n}|Dy_n| \leqslant |D|(b^{-n}|y_n|) \leqslant |D||\underline{y}|$.
– $\mathcal{C}$ is obtained by composing the automorphism $\underline{y} \mapsto (-Cy_n)_{n\in\mathbf{N}}$ of $\mathcal{K}_b$ and the endomorphism $\underline{y} \mapsto (y_n - C^{-1}y_{n+1})_{n\in\mathbf{N}}$ of $\mathcal{K}_b$, which is an automorphism of the form "Id + strict contraction" since we have

$$b^{-n}|C^{-1}y_{n+1}| \leqslant b|C^{-1}|\big(b^{-(n+1)}|y_{n+1}|\big) \leqslant b|C^{-1}||\underline{y}| \quad \text{and} \quad b|C^{-1}| < 1.$$

The rest of the proof of Lemma 3.1 is a little too long to be given here, but it only requires some care.

Why Φ_0 *is tangent to* 0. The graph of $D\Phi(0)$ is the kernel of $DF(0,\underline{0})$. By the proof of Lemma 3.1(iii), as $\mathcal{C}$ is invertible, this kernel consists of sequences $\underline{z}$ with $\underline{y} = \underline{0}$.

Remark 2. Our proof of Theorem 1.1 follows essentially the same lines as Hartman's in [7]: his first step is just a proof of the version of the pseudo-unstable manifold theorem needed, the main difference being in the second step, where we use the approach via invariant manifolds and the Perron–Irwin method initiated in [3]. This answers a question asked by Jürgen Moser in 1993 when Marc Chaperon gave a talk on [3] in Oberwolfach.

Apparently, it is not known whether every $C^{1+\mathrm{Lip}}$ strict contraction germ in an infinite dimensional space is C^1-linearizable.

This Note is an account of the first chapter of the author's doctoral dissertation [1].

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