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From classical to semiclassical non-trapping behaviour

Thierry Jecko

IRMAR, Université de Rennes I, campus Beaulieu, 35042 Rennes cedex, France

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Abstract

For the semiclassical Schrödinger operator with smooth long-range potential, we prove in a new way, making use of semiclassical measures, that the boundary values of its resolvent at non-trapping energies are bounded by $O(1/h)$, h being the semiclassical parameter. ***To cite this article: T. Jecko, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Non-capture : du classique au semi-classique. Pour l'opérateur de Schrödinger semi-classique avec potentiel lisse à longue portée, on montre d'une manière nouvelle, au moyen de mesures semi-classiques, que les valeurs au bord de sa résolvante aux énergies non-captives sont de taille $O(1/h)$, où h est le paramètre semi-classique. ***Pour citer cet article : T. Jecko, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Introduction

Concerning the Schrödinger operator with smooth long-range potential, it is well known that the boundary values of the resolvent at positive energies exist (cf. [2]). At the semiclassical level with respect to Planck's constant h, it is known that these boundary values at energy $\lambda > 0$ are $O(h^{-1})$ if and only if λ is non-trapping for the associated classical flow. While the necessity of the non-trapping condition was proved in [9], the bound $O(h^{-1})$ for the boundary values of the resolvent was derived from the non-trapping condition in [7,5] using a semiclassical version of Mourre's commutator method and in [8] (in greater generality) using a semiclassical Mourre estimate. For the latter result, an alternative approach was introduced by Burq in [1], in a general setting, but for compactly supported pertubation of the Laplacian. Our purpose here is to adapt Burq's approach for smooth long-range potentials.

Our main motivation concerns the corresponding result for matricial operators, for which there are difficulties to apply Mourre's theory. It is thus interesting to investigate another strategy in a simple and close framework. We

E-mail address: thierry.jecko@univ-rennes1.fr (T. Jecko).

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also think it is useful to present a rather short and elementary proof, based on pseudodifferential calculus together with the use of semiclassical measures.

Let us now introduce some notation and the result. We denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the natural norm and scalar product, respectively, in $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C})$ with $d\geqslant 1$ and by Δ_x the Laplacian in $\mathbb{R}^d$. Let $V\in C^\infty(\mathbb{R}^d;\mathbb{R})$, satisfying, for some $\rho>0$,

$$\forall\alpha\in\mathbb{N}^d,\ \forall x\in\mathbb{R}^d,\quad \left|\partial_x^\alpha V(x)\right|=\mathrm{O}_\alpha\big(\langle x\rangle^{-\rho-|\alpha|}\big), \tag{1}$$

where $\langle x\rangle=(1+|x|^2)^{1/2}$. Let $h\in]0;h_0]$, for some $h_0>0$. The semiclassical Schrödinger operator with smooth, long-range potential is given by $P_h:=-h^2\Delta_x+V(x)$, acting in $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C})$. It is well known that P_h is self-adjoint on the domain D of the Laplacian (see [2]). Denoting its resolvent by $R(z):=(P_h-z)^{-1}$, with z in the resolvent set of P_h, we know from [2] that it has boundary values $R(\lambda\pm\mathrm{i}0)$, for $\lambda\in]0;+\infty[$, as bounded operators from $\mathrm{L}^2_s(\mathbb{R}^d;\mathbb{C})$ to $\mathrm{L}^2_{-s}(\mathbb{R}^d;\mathbb{C})$, for $s>1/2$. Here $\mathrm{L}^2_s(\mathbb{R}^d;\mathbb{C})$ denotes the weighted L^2 space of mesurable, $\mathbb{C}$-valued functions f on $\mathbb{R}^d$ such that $x\mapsto\langle x\rangle^s f(x)$ belongs to $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C})$. We denote by $p(x,\xi):=|\xi|^2+V(x)$, $(x,\xi)\in T^*\mathbb{R}^d$, the symbol of P_h and by ϕ^t the associated Hamilton flow on $T^*\mathbb{R}^d$. An energy λ is non-trapping for p if

$$\forall(x,\xi)\in p^{-1}(\lambda),\quad \lim_{t\to-\infty}\left|\phi^t(x,\xi)\right|=+\infty\quad\text{and}\quad\lim_{t\to+\infty}\left|\phi^t(x,\xi)\right|=+\infty. \tag{2}$$

Our goal is to give a new proof of the following result, already obtained in [7,5,8].

Theorem 1.1. *Under the previous assumptions, let $I\subset]0;+\infty[$ be a compact interval of non-trapping energies for p. Then, for small enough h_0 and any $s>1/2$, there exists $C_s>0$ such that, uniformly for $\lambda\in I$ and $h\in]0;h_0]$,*

$$\left\|\langle x\rangle^{-s}R(\lambda\pm\mathrm{i}0)\langle x\rangle^{-s}\right\|\leqslant C_s h^{-1}. \tag{3}$$

2. Proof of Theorem 1.1: semiclassical trapping

Without assuming the non-trapping condition, we first study the situation, called semiclassical trapping, for which (3) is false. Then we show that this semiclassical trapping contradicts the non-trapping condition (2), yielding a proof of Theorem 1.1 by contradiction. The situation here is similar to that in [1], the strategy of which we follow. However, new ingredients and new results appear in the present Note.

Possibly after extraction of subsequences, the negation of (3) for some $s>1/2$ and for $R(\lambda+\mathrm{i}0)$ implies the existence of sequences $(h_n)_n\in]0;h_0]^{\mathbb{N}}$ tending to zero, $(f_n)_n$ of nonzero functions of the domain D, and $(z_n)_n\in\mathbb{C}^{\mathbb{N}}$ with $\Re(z_n)\to\lambda>0$ and $0\leqslant\Im(z_n)/h_n\to r\geqslant 0$, such that $\|\langle x\rangle^{-s}f_n\|=1$ and $\|\langle x\rangle^{s}(P_{h_n}-z_n)f_n\|=\mathrm{o}(h_n)$. We can also assume that the previous bounded sequence $(\langle x\rangle^{-s}f_n)_n$ in L^2 is pure and we denote by μ_s its semiclassical measure. It is a finite, non-negative Radon measure on $T^*\mathbb{R}^d$, satisfying, for any $a\in C_0^\infty(T^*\mathbb{R}^d)$,

$$\lim_{n\to\infty}\left\langle a^w_{h_n}\langle x\rangle^{-s}f_n,\langle x\rangle^{-s}f_n\right\rangle=\int_{T^*\mathbb{R}^d}a(x,\xi)\,\mu_s(\mathrm{d}x\,\mathrm{d}\xi)=:\mu_s(a). \tag{4}$$

Here a^w_h the Weyl h-quantization of the symbol a. Let us mention that we shall use at many places well known properties of semiclassical measures, of the Weyl h-pseudodifferential calculus, and of the functional calculus of Helffer–Sjöstrand. For details, we refer to [3,4,6].

Take $a\in C_0^\infty(T^*\mathbb{R}^d)$ with support disjoint from $p^{-1}(\lambda)$, the sequence $a\langle x\rangle^{-2s}(p-z_n)^{-1}\in C_0^\infty(T^*\mathbb{R}^d)$ is bounded, since $\Re(z_n)\to\lambda$. Therefore $\langle a^w_{h_n}\langle x\rangle^{-s}f_n,\langle x\rangle^{-s}f_n\rangle$ tends to 0 as $n\to\infty$, since $\|\langle x\rangle^{s}(P_{h_n}-z_n)f_n\|=\mathrm{o}(h_n)$. This means that μ_s is supported in $p^{-1}(\lambda)$.

According to [1], we expect that the Poisson bracket (in the distributional sense) $\{p,\langle x\rangle^{2s}\mu_s\}$ equals $r\langle x\rangle^{2s}\mu_s$. But it turns out that $r=0$ in our case. Indeed,

$$\begin{aligned}\mathrm{o}(h_n) &= \big\langle \langle x\rangle^s (P_{h_n} - z_n) f_n, \langle x\rangle^{-s} f_n\big\rangle = \big\langle (P_{h_n} - z_n) f_n, f_n\big\rangle = \big\langle f_n, (P_{h_n} - \overline{z_n}) f_n\big\rangle\\ &= \big\langle f_n, (P_{h_n} - z_n) f_n\big\rangle + \big\langle f_n, 2\mathrm{i}\Im(z_n) f_n\big\rangle = \mathrm{o}(h_n) + \big\langle f_n, 2\mathrm{i}\Im(z_n) f_n\big\rangle,\end{aligned}$$

yielding $\|f_n\|^2\Im(z_n)/h_n = \mathrm{o}(1)$. If $\lim \Im(z_n)/h_n = r \neq 0$ then $\|f_n\|^2 \to 0$. Since $s \geqslant 0$, $1 = \|\langle x\rangle^{-s} f_n\|^2 \leqslant \|f_n\|^2$. We arrive at a contradiction. Therefore $r = 0$ and we get the following result.

Proposition 2.1. *The measure $\langle x\rangle^{2s}\mu_s$ is invariant under ϕ^t, that is $\{p, \langle x\rangle^{2s}\mu_s\} = 0$.*

Proof. We follow [1]. For any $a \in C_0^\infty(T^*\mathbb{R}^d)$,

$$\begin{aligned}\big\langle \mathrm{i}h_n^{-1}\big[P_{h_n}, a^w_{h_n}\big] f_n, f_n\big\rangle &= \big\langle \mathrm{i}h_n^{-1} a^w_{h_n} f_n, (P_{h_n} - \overline{z_n}) f_n\big\rangle - \big\langle \mathrm{i}h_n^{-1} a^w_{h_n}(P_{h_n} - z_n) f_n, f_n\big\rangle\\ &= \big(2\Im(z_n)/h_n\big)\big\langle a^w_{h_n} f_n, f_n\big\rangle + \mathrm{o}(1) = \mathrm{o}(1),\\ \big\langle \mathrm{i}h_n^{-1}\big[P_{h_n}, a^w_{h_n}\big] f_n, f_n\big\rangle &= \big\langle \{p, a\}^w_{h_n} f_n, f_n\big\rangle + \mathrm{O}(h_n) = \mu_s\big(\langle x\rangle^{2s}\{p, a\}\big) + \mathrm{o}(1). \qquad \square\end{aligned} \tag{5}$$

Looking for other properties of μ_s, we learn from [4] that, if

$$\lim_{R\to+\infty} \limsup_{n\to\infty} \int\limits_{|x|>R} \langle x\rangle^{-2s} f_n^2 \,\mathrm{d}x = 0, \tag{6}$$

then the total mass of μ_s equals $\lim_{n\to\infty} \|\langle x\rangle^{-s} f_n\|^2$. In particular, μ_s is nonzero. From (1), we see that there exists $c > 0$ such that $\{p, x\cdot\xi\} \geqslant c$ on $p^{-1}(\lambda)$, for $|x|$ large enough. 'Quantizing' this fact carefully, in a similar way as in [8], we shall show a stronger version of (6), namely (10).

Let $R > 0$ and let $\mathbb{1}_{\{|x|>R\}}$ be the characteristic function of the set $\{(x, \xi) \in T^*\mathbb{R}^d;\ |x| > R\} =: T^*\mathbb{R}^d \setminus B^*_R$. Let $\chi_0 \in C_0^\infty(\mathbb{R};\mathbb{R})$ such that $0 \leqslant \chi_0 \leqslant 1$, $\chi_0 = 0$ on $]-\infty; 1/3[$, and $\chi_0 = 1$ on $]2/3; +\infty[$. For R large enough and any $\delta \in]0; \min(1;\rho)[$, where ρ appeared in (1), we can define, near $p^{-1}(\lambda) \setminus B^*_R$, the symbol $a_\infty(x,\xi) := \hat{x}\cdot\hat{\xi} - |x|^{-\delta}(\chi_0(\hat{x}\cdot\hat{\xi}) - \chi_0(-\hat{x}\cdot\hat{\xi}))$, where $\hat{x} := x/|x|$. It is easy to show that, near $p^{-1}(\lambda) \setminus B^*_R$, a_∞ is a smooth, bounded function such that, for $m = 0$,

$$\forall(\alpha,\beta) \in \mathbb{N}^{2d},\ \exists C_{\alpha\beta} > 0;\ \forall(x,\xi) \in p^{-1}(\lambda) \setminus B^*_R, \quad \big|\partial_x^\alpha \partial_\xi^\beta a_\infty(x,\xi)\big| \leqslant C_{\alpha\beta}\langle x\rangle^{-m-|\alpha|}. \tag{7}$$

Let $\tau \in C_0^\infty(\mathbb{R};\mathbb{R})$ with $0 \leqslant \tau \leqslant 1$, $\tau = 1$ on $[-R; R]$, $\operatorname{supp}\tau \subset [-R-1; R+1]$, and let $\chi(x) := \tau(|x|)$. For R large enough, there is, thanks to (1), some $c > 0$ such that $\{p, a_\infty\} \geqslant 2c\langle x\rangle^{-1-\delta}$ and $\{p, (1-\chi)^2\}a_\infty \geqslant 0$ near $p^{-1}(\lambda) \setminus B^*_R$. Let $\theta \in C_0^\infty(\mathbb{R};\mathbb{R}^+)$ with support sufficiently close to λ. By the Gårding inequality (cf. [3]),

$$\theta(P_{h_n})\big(\big\{p, (1-\chi)^2\big\}a_\infty\big)^w_{h_n}\theta(P_{h_n}) \geqslant \langle x\rangle^{-m}\theta(P_{h_n})\widetilde{O}(h_n)\theta(P_{h_n})\langle x\rangle^{-m}, \tag{8}$$

for any $m \in \mathbb{N}$, where $\widetilde{O}_m(h_n)$ denotes a bounded operator, the norm of which is $O_m(h_n)$. By the h-pseudo-differential calculus, for symbols satisfying (7) with $m = 1$,

$$\begin{aligned}\theta(P_{h_n})(1-\chi)\big\{p, a_\infty\big\}^w_{h_n}(1-\chi)\theta(P_{h_n}) &\geqslant c\langle x\rangle^{-(1+\delta)/2}\theta(P_{h_n})(1-\chi)^2\theta(P_{h_n})\langle x\rangle^{-(1+\delta)/2}\\ &\quad + \langle x\rangle^{-1}\theta(P_{h_n})\widetilde{O}(h_n)\theta(P_{h_n})\langle x\rangle^{-1}.\end{aligned} \tag{9}$$

Since $\mathrm{i}h_n^{-1}[P_{h_n}, ((1-\chi)^2 a_\infty)^w_{h_n}] = \{p, (1-\chi)^2 a_\infty\}^w_{h_n} + \langle x\rangle^{-1}\widetilde{O}(h_n)\langle x\rangle^{-1}$ and $\|\tilde{\chi}\theta(P_{h_n})\langle x\rangle^{s-1}\langle x\rangle^{-s} f_n\| = \mathrm{O}(1)$ for $\tilde{\chi} \in C_0^\infty(\mathbb{R}^d)$, we obtain, using $\delta \leqslant 1$, (8), and (9),

$$\big\langle \mathrm{i}h_n^{-1}\big[P_{h_n}, \big((1-\chi)^2 a_\infty\big)^w_{h_n}\big]\theta(P_{h_n}) f_n, \theta(P_{h_n}) f_n\big\rangle \geqslant (c/2)\big\|(1-\chi)\theta(P_{h_n})\langle x\rangle^{-(1+\delta)/2} f_n\big\|^2 + \mathrm{O}(h_n).$$

Since a_∞ is a bounded symbol, the l.h.s. of the last inequality tends to zero as in (5), yielding

$$\lim_{n\to\infty} \big\|\mathbb{1}_{\{|x|>R\}}\theta(P_{h_n})\langle x\rangle^{-(1+\delta)/2} f_n\big\|^2 = 0. \tag{10}$$

Notice that this and the energy localization of the f_n show that $(\langle x\rangle^{-(1+\delta)/2} f_n)_n$ is bounded in L^2. It is also pure and its semiclassical measure $\mu_{(1+\delta)/2}$ satisfies, for $a \in C_0^\infty(T^*\mathbb{R}^d)$, $\mu_{(1+\delta)/2}(a) = \mu_s(\langle x\rangle^{2s-1-\delta} a)$. The limit (10) implies that $\mu_{(1+\delta)/2}$ is supported in B_R^*. Thus μ_s have compact support included in $p^{-1}(\lambda) \cap B_R^*$ and (6) holds true for the sequence $(\langle x\rangle^{-(1+\delta)/2} f_n)_n$. Therefore, this sequence converges to the total mass of $\mu_{(1+\delta)/2}$. Since $s > 1/2$, we can choose $\delta \leqslant s$, yielding $\|\langle x\rangle^{-(1+\delta)/2} f_n\|^2 \geqslant \|\langle x\rangle^{-s} f_n\|^2 = 1$, for all n. This shows that μ_s is nonzero.

Now, assume that λ is a non-trapping energy. Since μ_s has compact support in $p^{-1}(\lambda) \cap B_R^*$, we can find $g \in C_0^\infty(T^*\mathbb{R}^d)$ with $g = 1$ on $p^{-1}(\lambda) \cap B_R^*$. By the non-trapping condition (2), $a(x,\xi) := -\int_0^{+\infty} g \circ \phi^t(x,\xi)\,\mathrm{d}t$ is a well-defined, smooth function near $p^{-1}(\lambda)$ such that $\{p, a\} = g$. Since μ_s has compact support, $\mu_s(\langle x\rangle^{2s}) = \mu_s(\langle x\rangle^{2s}\{p, a\}) = 0$, by Proposition 2.1, leading to a contradiction. This ends the proof of Theorem 1.1.

What would happen, if $s \leqslant 1/2$? If λ is non-trapping, $\mu_s = 0$ and the previous arguments show that $\lim_{n\to\infty} \|\langle x\rangle^{-(1+\delta)/2} f_n\|^2 = 0$ which does not contradict a priori $\|\langle x\rangle^{-s} f_n\| = 1$. This appears for $V = 0$, for which each $\lambda > 0$ is non-trapping, and for $s \in [0; 1/2[$: given $k \in \mathbb{R}^d \setminus \{0\}$, $\chi \in C_0^\infty(]1; 2[; \mathbb{R})$ with $\int_{\mathbb{R}} \chi^2 = 1$, and denoting by m_d the Lebesgue measure of the $(d-1)$-dimensional unit sphere, take

$$f_n(x) := \frac{1}{\sqrt{m_d}}\, \mathrm{e}^{\mathrm{i}h_n^{-1}k\cdot x} \langle x\rangle^s |x|^{(1-d)/2} \frac{1}{\sqrt{n}}\, \chi\big(|x|/n\big).$$

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