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Partial Differential Equations

New estimates for the Laplacian, the div–curl, and related Hodge systems

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Abstract

We establish new estimates for the Laplacian, the div–curl system, and more general Hodge systems in arbitrary dimension, with an application to minimizers of the Ginzburg–Landau energy. ***To cite this article: J. Bourgain, H. Brezis, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Nouvelles estimées pour le Laplacien, le système div–rot et autres systèmes de Hodge. On établit de nouvelles estimées pour le Laplacien, le système div–rot et autres systèmes de Hodge en dimension quelconque. On présente une application aux minimiseurs de l'énergie de Ginzburg–Landau. ***Pour citer cet article : J. Bourgain, H. Brezis, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

On démontre que l'équation rot $Y = g$, où g est un champ vectoriel $g \in L^3(\mathbb{R}^3, \mathbb{R}^3)$, à divergence nulle, possède une solution Y dans L^∞ avec ∇Y dans L^3. On en déduit, en particulier, l'inégalité

$$\left\| \nabla \left(\frac{1}{|x|} * f \right) \right\|_{3/2} \leqslant c \|f\|_1$$

pour tout $f \in L^1(\mathbb{R}^3, \mathbb{R}^3)$, avec div $f = 0$.

Ces résultats se généralisent au cadre de Hodge pour les formes différentielles en dimension arbitraire. On indique une application à des questions de régularité optimale pour les minimiseurs de l'énergie de Ginzburg–Landau.

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The starting point for this work is the following estimate from [5, Proposition 4].

Theorem 1. *Let Γ be a closed rectifiable curve in $\mathbb{R}^3$ with unit tangent vector t and let $Y \in W^{1,3}(\mathbb{R}^3, \mathbb{R}^3)$. Then*

$$\left|\int_\Gamma Yt\right| \leqslant C|\Gamma|\|\nabla Y\|_3. \tag{1}$$

The proof in [5] relies on a Littlewood–Paley decomposition and another proof was given recently by Van Schaftingen [8] which uses only the Morrey–Sobolev imbedding.

Remark 1. The same proof as in [5] or [8] yields a similar inequality for any fractional Sobolev norm $W^{s,p}$, with $sp = 3$ and

$$|Y|^p_{W^{s,p}} = \iint \frac{|Y(x) - Y(y)|^p}{|x-y|^6}, \quad p > 3,$$

in place of $\|\nabla Y\|_3$.

Here is a simple estimate for the div–curl system of the type studied in this Note. Consider in $\mathbb{R}^3$ the system

$$\begin{cases} \operatorname{curl} Z = f, \\ \operatorname{div} Z = 0 \end{cases} \tag{2}$$

for a given divergence-free vector field f. It is standard that this system has a solution, namely

$$Z = (-\Delta)^{-1} \operatorname{curl} f.$$

The standard Calderon–Zygmund theory implies that

$$\|\nabla Z\|_p \leqslant C_p\|f\|_p, \quad 1 < p < \infty. \tag{3}$$

Consequently (via the Sobolev imbedding) we have for $1 < p < 3$,

$$\|Z\|_{p^*} \leqslant C_p\|f\|_p, \quad 1 < p < \infty \tag{4}$$

with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$.

We now turn to the case $p = 1$. One may easily see that (3) fails for $p = 1$. Surprisingly, (4) survives for $p = 1$.

Theorem 2. *We have*

$$\|Z\|_{3/2} \leqslant C\|f\|_1. \tag{5}$$

Theorem 2 implies Theorem 1. Indeed, consider the vector-field

$$f = |\Gamma|^{-1}\mathcal{H}_\Gamma t,$$

where $\mathcal{H}_\Gamma$ is the 1-dimensional Hausdorff measure on Γ. Clearly $\operatorname{div} f = 0$. Solve (2) for this f; the corresponding Z satisfies

$$\|Z\|_{3/2} \leqslant C$$

(here we have ignored the fact that f is not an L^1 function, but is a measure). Next, write

$$|\Gamma|^{-1}\int_\Gamma Yt = \int_{\mathbb{R}^3} Y \operatorname{curl} Z$$

and thus

$$|\Gamma|^{-1}\left|\int\limits_{\Gamma} Yt\right| \leqslant \|Z\|_{3/2}\|\operatorname{curl} Y\|_3,$$

which yields (1).

One may also derive Theorem 2 from Theorem 1 using Smirnov's theorem [7] which asserts that every

$$f \in L^1_\# = \{f \in L^1; \operatorname{div} f = 0\}$$

may be written as a weak limit (in the sense of measures) of combinations of the form

$$\sum \alpha_i \frac{1}{|\Gamma_i|}\mathcal{H}_{\Gamma_i} t_i$$

with $\alpha_i \geqslant 0\ \forall i$ and $\sum \alpha_i \leqslant \|f\|_1$.

From this fact and Theorem 1 we obtain

$$\left|\int Yf\right| \leqslant C\|f\|_1\|\nabla Y\|_3$$

for every $f \in L^1_\#$.

By Hahn–Banach, this means that for every $Y \in W^{1,3}$, $\operatorname{curl} Y = \operatorname{curl} Y'$ (in the distributional sense) for some Y' controlled in L^∞ (by $\|\nabla Y\|_3$). Theorem 2 follows by duality and a Hodge decomposition.

Remark 2. It should be pointed out that the analogue of Theorem 2 for $n = 2$ fails. Indeed take $Z = (-x_2/|x|^2, x_1/|x|^2)$ for which $\operatorname{curl} Z = 2\pi\delta_0$, $\operatorname{div} Z = 0$, while Z is not L^2.

There is another approach to Theorem 2 via an explicit (but nonlinear) constructive way of obtaining Y'.

Theorem 3. *Given $g \in L^3_\#(\mathbb{R}^3, \mathbb{R}^3)$ there exists $Y \in C^0 \cap W^{1,3} \cap L^\infty$ satisfying*

$$\operatorname{curl} Y = g \tag{6}$$

and

$$\|Y\|_\infty + \|\nabla Y\|_3 \leqslant C\|g\|_3. \tag{7}$$

Here and throughout the rest of this Note $W^{1,p}$ denotes the completion of C_0^∞ with respect to the norm $\|\nabla f\|_p$.

Theorem 3 resembles (and in fact implies) a result we established in [3] for the divergence equation. In the same way as in [3] one can show that there is no bounded operator $T : L^3_\# \to L^\infty$ satisfying $\operatorname{curl} T = \mathrm{Id}$.

Remark 3. Theorem 3 is stronger than Theorem 2. By duality it is equivalent to a refined version of the theorem where (5) is replaced by

$$\|Z\|_{3/2} \leqslant C\|f\|_{L^1+W^{-1,3/2}}. \tag{5'}$$

Another assertion (equivalent to Theorem 2) is

Corollary 1. *We have, for every $f \in L^1_\#(\mathbb{R}^3, \mathbb{R}^3)$,*

$$\left\|\nabla\left(\frac{1}{|x|} * f\right)\right\|_{3/2} \leqslant C\|f\|_1. \tag{8}$$

Thus, consequently, for every $f \in L^1_\#(\mathbb{R}^3, \mathbb{R}^3)$,

$$\left\| \frac{1}{|x|} * f \right\|_3 \leqslant C\|f\|_1. \tag{9}$$

Remark 4. A 'natural' inequality stronger than (8), involving second-order derivatives,

$$\left\| \nabla^2 \left(\frac{1}{|x|} * f \right) \right\|_1 \leqslant C\|f\|_1 \tag{8'}$$

is *not* true.

Remark 5. There are inequalities similar to (8) and (9) in 2-d: for every $f \in L^1_\#(\mathbb{R}^2, \mathbb{R}^2)$,

$$\left\| \nabla \left(\log \frac{1}{|x|} * f \right) \right\|_2 \leqslant C\|f\|_1 \quad \text{and} \quad \left\| \log \frac{1}{|x|} * f \right\|_\infty \leqslant C\|f\|_1.$$

Remark 6. A stronger form of Corollary 1 asserts that, for every $f \in (L^1 + W^{-1,3/2})_\#$, (8) and (9) hold with $\|f\|_1$ being replaced by $\|f\|_{L^1+W^{-1,3/2}}$.

Corollary 2. *Every $f \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ admits a decomposition*

$$f = \operatorname{curl} Y + \operatorname{grad} P$$

with $Y \in W^{1,3} \cap L^\infty$, $P \in W^{1,3}$.

The preceding has a generalization to Hodge-type systems in arbitrary dimension. Denote Λ^ℓ the space of ℓ-forms on $\mathbb{R}^n$ $(0 \leqslant \ell \leqslant n)$. There is the following extension of Theorem 3

Theorem 4. *For every $0 < \ell \leqslant n-1$, we have that*

$$\mathrm{d}W^{1,n}(\Lambda^\ell) = \mathrm{d}(W^{1,n} \cap L^\infty)(\Lambda^\ell).$$

Here d denotes the exterior differential operator; see [6] for the notations. Notice that for $\ell = 0$, the statement obviously fails (grad f for $f \in W^{1,n}$ is not necessarily equal to grad g for some $g \in L^\infty$). Also in $n = 3$, the div-theorem from [3] corresponds to the case $\ell = 2$, and Theorem 3 above to $\ell = 1$.

There is, in particular, the following corollary of Theorem 4 to Hodge-decompositions generalizing Corollary 2. (We state the result here on a domain M, say a cube for simplicity.)

Corollary 3. *Let $n \geqslant 3$. Then*

$$L^n(\Lambda^1 M) = \mathrm{d}W_0^{1,n}(\Lambda^0 M) \oplus \mathrm{d}^*(W^{1,n} \cap L^\infty)(\Lambda^2 M).$$

This fact is an important ingredient in the proof of

Theorem 5. *Assume $n \geqslant 3$ and fix a boundary condition $g \in H^{1/2}(\partial M, S^1)$. Let u_ε be a minimizer of the Ginzburg–Landau energy*

$$E_\varepsilon(u) = \frac{1}{2}\int_M |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_M \left(|u|^2 - 1\right)^2$$

in the class $\{u \in H^1(M, \mathbb{C});\ u = g \text{ on } \partial M\}$. Then

$$\|u_\varepsilon\|_{W^{1,n/(n-1)}} \leqslant C \quad \text{as } \varepsilon \to 0.$$

For $n = 3$, this has been established in [1,2] following an earlier argument from [5]. Corollary 3 permits us to generalize the argument in [1,2] to general dimension $n > 3$. For $n = 2$, Corollary 3 (and its consequence for Ginzburg–Landau) fails.

A word about proofs. The key analytical ingredient to obtaining Theorem 4 is the following:

Theorem 6. *For all $\delta \stackrel{>}{\to} 0$, there is a constant C_δ such that if $f \in W^{1,n}(\mathbb{R}^n)$, $\|f\|_{1,n} \leqslant 1$ and we fix one of the variables $i = 1, \ldots, n$, there exists $g \in (W^{1,n} \cap L^\infty)(\mathbb{R}^n)$ satisfying*

(i) $\|g\|_{1,n} + \|g\|_\infty \leqslant C_\delta$,
(ii) $\max_{j \neq i} \|\partial_j(f - g)\|_n < \delta$.

Notice that one may not take the full gradient in (ii), since that clearly would imply that $f \in L^\infty$.

The argument is constructive and starts with a Littlewood–Paley decomposition.

Using Theorem 6, one may extend Theorem 2 concerning (2) to rather general first order elliptic systems (see [4]).

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