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Standing waves on infinite depth

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Abstract

The two-dimensional standing wave problem, for an infinitely deep layer, is considered, based on the formulation of the problem as a second order non local PDE. Despite the presence of infinitely many resonances in the linearized problem, we use the Nash–Moser implicit function theorem to prove the existence of standing waves corresponding to values of the amplitude ε having 0 as a Lebesgue point. ***To cite this article: G. Iooss et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Ondes de gravité stationnaires en profondeur infinie. On considère le problème des ondes de gravité stationnaires (le clapotis) en profondeur infinie, mis sous la forme d'une EDP du second ordre non locale. Malgré la dégénérescence infinie du problème linéarisé, nous adaptons le théorème des fonctions implicites de Nash–Moser pour montrer l'existence de vagues stationnaires pour un ensemble de valeurs de l'amplitude ε ayant 0 comme point de Lebesgue. ***Pour citer cet article : G. Iooss et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

On s'intéresse à l'écoulement potentiel bidimensionnel des ondes de gravité stationnaires (le clapotis) à l'aide de la formulation (4) qui fait intervenir une EDP du second ordre $\mathcal{F}(w,\mu)=0$ non locale vérifiée par la fonction $w(x,t)$, périodique et paire en x et t. Il y a un unique paramètre μ. Ce paramètre fait intervenir la gravité, la période temporelle et la longueur d'onde dans la direction horizontale. Dans l'Éq. (4) $\mathcal{H}$ désigne la transformée de Hilbert périodique (en variable d'espace), un point ou un accent désigne respectivement une dérivée partielle par rapport à t ou à x.

La linéarisation en $w=0$ permet de définir un opérateur linéaire $\mathcal{L}_\mu$ dont le noyau est de dimension infinie pour toute valeur rationnelle de μ (voir le Lemme 1.1). Choisissant la valeur critique 1 pour μ, on sait grâce à [2] et

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[5] qu'il y a une infinité de solutions approchées sous la forme de développements en séries $w_\varepsilon^{(N)}$ de puissances de l'amplitude ε des vagues, avec $\mu = 1 + \varepsilon^2/4$, dont la forme est donnée au Lemme 1.2. Le pseudo-inverse de $\mathcal{L}_1$ est borné, non régularisant dans tout $H_{\natural\natural}^{m,ee}$ (défini en (5)), alors que les termes non linéaires de $\mathcal{F}$ contiennent des dérivées du second ordre en x et t. Il est alors nécessaire d'adapter le théorème des fonctions implicites de Nash–Moser pour obtenir un résultat d'existence pour les solutions correspondant aux $w_\varepsilon^{(N)}$ précédents. Le principal résultat de cette Note est le théorème suivant

Théorème 0.1. *Pour m assez grand, $p \geqslant 4, N \geqslant p+1$, et $\varepsilon_0 > 0$ assez petit, il existe un ensemble mesurable $\mathcal{E} \subset (0, \varepsilon_0)$, avec 0 comme point de Lebesgue (voir (17)), tel que pour tout $\varepsilon \in \mathcal{E}$ et $\mu = 1 + \varepsilon^2/4$, il existe une solution w de $\mathcal{F}(w, \mu) = 0$ dans l'espace $H_{\natural\natural}^{m,ee}$ dont le développement asymptotique à l'ordre $p-1$ en ε est donné par $w_\varepsilon^{(p-1)}$, avec $w^{(1)} = \cos x \cos t$.*

Remarque. On conjecture que ce théorème est encore vrai pour toute autre solution correspondant à l'un des $w_\varepsilon^{(N)}$ donnés au Lemme 1.2, mais la démonstration nécessite encore du travail.

La principale difficulté pour appliquer la méthode de Newton utilisée dans le théorème de Nash–Moser est de savoir inverser l'opérateur linéarisé en tout point $w \neq 0$. L'opérateur linéaire obtenu donne une EDP du second ordre non locale, à coefficients périodiques, voisine de celle, hyperbolique, correspondant à $\mathcal{L}_1$, mais contenant en plus des dérivées du second ordre en xx et en xt. Le pas fondamental permettant d'utiliser la méthode développée par Plotnikov et Toland dans [7], est fourni par le Théorème 2.1, qui montre qu'après un changement de variables adéquat, l'équation linéarisée pour la nouvelle variable φ s'écrit sous la forme (10) où n'interviennent que le seul terme $\partial_{tt}\varphi$ avec une dérivée partielle du second ordre, un terme non local $\partial_y\{q\mathcal{H}\varphi\}$ avec une dérivée du premier ordre en y (nouvelle coordonnée d'espace), un terme $\mathcal{G}(\varphi)$ quasi uni-dimensionnel (Q1D) comme défini dans [7] (régularisant en y), et un terme $c(\mathcal{F}, \varphi)$ s'annulant lorsqu'on linéarise en un point w solution de $\mathcal{F}(w, \mu) = 0$.

Ensuite, la méthode de moyennisation de [7] (voir le Théorème 3.1) fait apparaître deux coefficients constants δ et κ dépendant du point de linéarisation w, l'opérateur linéaire à inverser étant la somme d'un opérateur $\mathcal{A}$, diagonal en composantes de Fourier, et d'un opérateur $-\mathcal{B}$ régularisant dans un sens approprié, spécifié au Théorème 2.2 (voir 13). Considérant maintenant le cas où le développement de w commence par $\varepsilon \cos x \cos t$ avec $\mu = 1 + \varepsilon^2/4$ on montre que l'inversibilité de $\mathcal{A} - \mathcal{B}$ repose d'une part sur l'inversibilité de la restriction $\mathcal{M}_\varepsilon$ de $\mathcal{A} - \mathcal{B}$ au noyau de $\mathcal{L}_1$, d'autre part sur l'étude de l'inverse de la restriction $\Lambda_\varepsilon^{(0)}$ de $\mathcal{A} - \mathcal{B}$ au supplémentaire du noyau, opérateur qui fait intervenir *un problème de petits diviseurs*. On impose alors une condition diophantienne (16) sur (δ, κ) que l'on contrôle au long de l'itération de la méthode de Newton. Cela aboutit à ne montrer l'existence des ondes stationnaires que pour $\varepsilon \in \mathcal{E}$ où 0 est point de Lebesgue (17).

1. Formulation of the problem and the existence result

We consider an infinitely deep layer of perfect incompressible fluid in two-dimensional potential motion under gravity with a free surface without surface tension. We are interested in solutions which are periodic in time and in the horizontal direction, and invariant under reflection in the vertical axis. The existence of solutions in the finite depth case was proved recently by Plotnikov and Toland in [7]. The problem we consider below has the additional difficulty of being infinitely degenerate at the linearized level, as was noticed a long time ago (see, for instance, Poisson [8], Cauchy [3], Airy [1], Stokes [9]).

The formulation we take for the standing wave problem was introduced by Dyachenko et al. [4], where (w, ϕ) is a pair of unknown functions of (x, t), both even and 2π-periodic in x, and w and $\partial\phi/\partial t$ being even and 2π-periodic in t. In the following we denote by $\mathcal{H}$ the spatial periodic Hilbert transform, such that

$$\mathcal{H}\cos nx = -\sin|n|x, \quad \mathcal{H}\sin nx = sgn(n)\cos nx \quad (n \neq 0). \tag{1}$$

The system satisfied by (w, ϕ) can be written as follows (see [4] and [6])

$$(1+\mathcal{H}w')\dot{w} - w'\mathcal{H}\dot{w} - \mathcal{H}\phi' = 0, \tag{2}$$

$$(1+\mathcal{H}w')(\dot{\phi} - \mu w) - \phi'\mathcal{H}\dot{w} + \mathcal{H}\big(w'(\dot{\phi}-\mu w) - \dot{w}\phi'\big) = 0, \tag{3}$$

where a dot means a partial time (t) derivative, and a prime means a partial space (x) derivative. With these equations, the free surface is given parametrically in physical coordinates (ξ,η) by

$$(\xi,\eta) = \big(x + \mathcal{H}w(x,t), -w(x,t)\big), \quad (x,t)\in\mathbb{R}^2.$$

The function $\phi(x,t)$ is the value of the velocity potential on the free surface, while μ is the bifurcation parameter, $\mu = gT^2/(2\pi\lambda)$ where T is the time period, λ is the spatial period, and g is the gravitational acceleration. It is shown in [6] that the formulation (2), (3) is equivalent to the classical formulation for sufficiently smooth solutions.

Let us define the basic (Sobolev) spaces $H^m_{\natural\natural} = H^m\{(\mathbb{R}/2\pi\mathbb{Z})^2\}$, then Eqs. (2), (3) are equivalent to the *second order nonlocal PDE*:

$$\mathcal{F}(w,\mu) =: \partial_t(L_{w'}\dot{w}) - \mu\mathcal{H}w' + \mathcal{H}\partial_x\left\{\frac{1}{D}\mathcal{H}(L_{w'}\dot{w}\mathcal{H}L_{w'}\dot{w}) + (\mathcal{H}L_{w'}\dot{w})\mathcal{H}\left(\frac{1}{D}L_{w'}\dot{w}\right)\right\} = 0, \tag{4}$$

where

$$L_{w'}f = (1+\mathcal{H}w')f - w'\mathcal{H}f, \qquad D = (1+\mathcal{H}w')^2 + w'^2.$$

Notice that a constant may be added to a solution to yield another solution. Therefore there is no loss of generality in seeking solutions of (4) with zero mean with respect to space and time together.

Let us define for $m \geqslant 0$ suitable spaces for the study of this nonlinear system:

$$H^{m,ee}_{\natural\natural} = \big\{w\in H^m_{\natural\natural};\ w \text{ is even in } x \text{ and in } t,\ w \text{ has zero average}\big\}, \tag{5}$$

$$H^{m,eo}_{\natural\natural} = \big\{\phi\in H^m_{\natural\natural};\ \phi \text{ is even in } x \text{ and odd in } t\big\},$$

and so on, depending on evenness or oddness with respect to x and to t. Then, for $m \geqslant 3$, $\mathcal{F}$ is an analytic map from $H^{m,ee}_{\natural\natural}\times\mathbb{R}$ to $H^{m-2,ee}_{\natural\natural}$.

The linearization of (4) at the origin is

$$\mathcal{L}_\mu u =: D_w\mathcal{F}(0,\mu)u = \ddot{u} - \mu\mathcal{H}\partial_x u.$$

We claim the following

Lemma 1.1. *If $\mu\notin\mathbb{Q}$ the kernel of $\mathcal{L}_\mu$ in $H^{m,ee}_{\natural\natural}$ is $\{0\}$. For $\mu = 1$ the kernel of $\mathcal{L}_1$ in $H^{m,ee}_{\natural\natural}$ is the subspace $E_0\cap H^{m,ee}_{\natural\natural}$, with*

$$E_0 = \operatorname{span}\big\{A_q\cos q^2x\cos qt;\ A_q\in\mathbb{R},\ q\in\mathbb{N}\big\}.$$

For other rational values of μ, the kernel of $\mathcal{L}_\mu$ is infinite dimensional, and is easily deduced from E_0. For $f\in H^{m,ee}_{\natural\natural}$ which is orthogonal to E_0 in $L^2_{\natural\natural}$, there is a unique $u\in H^{m,ee}_{\natural\natural}$, orthogonal to E_0 in $L^2_{\natural\natural}$, which is solution of $\mathcal{L}_1u = f$.

Since the pseudo-inverse of $\mathcal{L}_1$ is not regularizing and since the nonlinear terms in (4) lose two degrees of regularity, we need to adapt the Nash–Moser implicit function theorem for the bifurcation problem.

Now, it is known that approximate solutions of the standing wave problem exist up to an arbitrary power of ε, where $\mu = 1+\varepsilon^2/4$. We have the following [5]

Lemma 1.2. *There are infinitely many approximate solutions w of* (4) *under the form of asymptotic expansions in powers of the amplitude ε of the wave*

$$w^{(N)}_\varepsilon = \sum_{1\leqslant n\leqslant N}\varepsilon^n w^{(n)}, \quad \mu = 1+\frac{\varepsilon^2}{4}, \quad w^{(1)} = \sum_{q\in I}\frac{\varepsilon_q}{q^2}\cos q^2x\cos qt, \quad \varepsilon_q = \pm 1,\ I\subset\mathbb{N},$$

with any finite subset I of $\mathbb{N}$, moreover $w_\varepsilon^{(N)}$ is such that

$$\mathcal{F}\left(w_\varepsilon^{(N)}, 1+\frac{\varepsilon^2}{4}\right) = \varepsilon^{N+1} Q_\varepsilon,$$

where Q_ε is bounded in any $H^{m,ee}_{\natural\natural}$ when $\varepsilon \to 0$ (see [2] *for $I = \{1\}$,* [5] *for the general result).*

The main result of this Note is the following

Theorem 1.3. *For m large enough, $p \geqslant 4$, $N \geqslant p+1$ and $\varepsilon_0 > 0$ small enough, there is a measurable set $\mathcal{E} \subset (0, \varepsilon_0)$, with 0 as a Lebesgue point (see* (17)*), such that for any $\varepsilon \in \mathcal{E}$ and $\mu = 1 + \varepsilon^2/4$, there exists a solution w of $\mathcal{F}(w, \mu) = 0$ in $H^{m,ee}_{\natural\natural}$, whose asymptotic expansion is $w_\varepsilon^{(N)}$, with $w^{(1)} = \cos x \cos t$, $\|w - w_\varepsilon^{(N)}\|_{H^m} = \mathrm{O}(\varepsilon^p)$.*

Remark 1. Same theorem holds when we take any rational value of μ, instead of 1.

Remark 2. We conjecture that the above theorem also holds for any other type of asymptotic form $w_\varepsilon^{(N)}$ of the solution w, indicated at Lemma 1.2. Notice that in this case the set $\mathcal{E}$ of "good values" of ε may depend on the form of $w^{(1)}$.

We give below a sketch of the quite technical proof of this theorem. The basic point, which is the key when using the Nash–Moser implicit function theorem, is that we can control the inverses of linearized operators coming from (4) when w is small and ε lies in a carefully-chosen parameter set $\mathcal{E}$.

2. Linearized operator at a nonzero point

Since there are infinitely many formal solutions, we specialize by looking for w in the form

$$w = w_\varepsilon^{(N)} + \varepsilon^p \underline{w}, \quad p \geqslant 4,\ N \geqslant p+1, \quad \mu = 1 + \varepsilon^2/4, \tag{6}$$

where $\underline{w}$ is of order 1. The equation to solve is then

$$\underline{\mathcal{F}}(\underline{w}, \varepsilon) =: \varepsilon^{-p} \mathcal{F}\left(w_\varepsilon^{(N)} + \varepsilon^p \underline{w}, 1 + \varepsilon^2/4\right) = 0, \tag{7}$$

where $\underline{\mathcal{F}}(0, \varepsilon) = \varepsilon^{N+1-p} Q_\varepsilon$. For $\underline{w} \in H^{m,ee}_{\natural\natural} \neq 0$, $m \geqslant 4$ and $0 \leqslant \varepsilon \leqslant \varepsilon_0$ small enough, let consider the linear equation

$$\partial_{\underline{w}} \underline{\mathcal{F}}(\underline{w}, \varepsilon) u = f, \tag{8}$$

and make the change of variable $v = L_{w'} u$ for the perturbation u of $\underline{w}$. It is then possible to write (8) as

$$\partial_t\left(\dot{v} - \partial_x(av)\right) + \mathcal{H}\partial_x\left\{a\mathcal{H}\left(\dot{v} - \partial_x(av)\right)\right\} - \mathcal{H}\partial_x\left\{(\mu - b)v\right\} + \Gamma(\mathcal{F}, v) = f \tag{9}$$

with

$$a = \mathcal{H}\left(\frac{1}{D} L_{w'} \dot{w}\right) + \frac{1}{D}\mathcal{H}(L_{w'}\dot{w}),$$
$$b = D^{-1}\left\{a^2 L_{w'} w'' - 2a L_{w'} \dot{w}' + L_{w'} \ddot{w} + \mu(D - 1 - \mathcal{H}w') + L_{w'} w''\left(D^{-2}\right)(\pi_0 L_{w'} \dot{w})^2\right\},$$
$$\Gamma(\mathcal{F}, v) = -\mathcal{H}\partial_x\left\{\mathcal{H}(\mathcal{F})\mathcal{H}\left(\frac{v}{D}\right)\right\} + \mathcal{H}\partial_x\left\{\frac{1}{D}\pi_0\dot{v} - \frac{1}{D}\mathcal{H}\partial_x\left(\frac{v}{D}\right)(\pi_0 L_{w'}\dot{w})\right\}\int_0^t \pi_0(\mathcal{F})\,\mathrm{d}s,$$

where we denote by π_0 the average with respect to x, and $\Gamma(\mathcal{F}, v)$ vanishes when w is a solution of $\mathcal{F}(w,\mu)=0$. The form of (9) suggests a change of coordinates to remove terms which contain second order derivatives in xx and in xt, except for those which involve $\mathcal{F}(w,\mu)$. Such a change of coordinates is defined by

$$y = x + d(x,t) =: \mathcal{U}_t(x), \qquad \partial_t d = a(1+\partial_x d), \qquad d|_{t=0} = 0.$$

From now on a tilde (~) means that a function of (x,t) is expressed in coordinates (y,t), and a hat (^) will indicate a function of (y,t) expressed as a function of (x,t), via the formulae

$$\tilde{u}(y,t) = u\big(\mathcal{U}_t^{-1}(y), t\big), \qquad \hat{v}(x,t) = v\big(\mathcal{U}_t(x), t\big).$$

Then we have the identity

$$\big\{\dot{v} - \partial_x(av)\big\}^{\sim} = p^{-1}\partial_t\{p\tilde{v}\},$$

where we define the new coefficient p by

$$p(y,t) = \mathrm{e}^{-\int_0^t \tilde{a}'(y,\tau)\,\mathrm{d}\tau} = 1 - \partial_y \tilde{d}(y,t).$$

Now we introduce the linear operators $\mathcal{S}_\omega$ and $\mathcal{S}$, defined for $\omega \in H^m_{\natural\natural}$, $m \geqslant 3$, and $w \in H^m_{\natural\natural}$, by the formulae

$$\mathcal{S}_\omega u = \mathcal{H}(\omega u) - \omega\mathcal{H}u, \qquad \mathcal{S}u = \widetilde{\big(\mathcal{H}(\hat{u})\big)} - \mathcal{H}u.$$

Both operators have *smoothing properties*. More precisely they are quasi one dimensional (Q1D) operators of order $(0,\beta)$ on H^s if $\beta + s \leqslant m-3$ for $\mathcal{S}_\omega$, and if $\beta + s \leqslant m-5$ for $\mathcal{S}$. The definition of Q1D operators follows [7] (see Definition 7.6).

The change of coordinates defined above kills the second order derivatives in xx and xt. More precisely, we show the following

Theorem 2.1. *Assume* $\underline{w} \in H^{m,ee}_{\natural\natural} \neq 0$, $m \geqslant 7$, *and* $0 \leqslant \varepsilon \leqslant \varepsilon_0$. *Then consider the linearized equation* $\partial_{\underline{w}}\underline{\mathcal{F}}(\underline{w},\varepsilon)u = f$, *and make the change of variable* $\varphi = p\widetilde{L_{w'}u}$. *Then* $\varphi(y,t) \in H^{k,ee}_{\natural\natural}$, $k \leqslant m-3$, *satisfies*

$$\partial_{tt}\varphi - \partial_y\{q\mathcal{H}\varphi\} + \mathcal{G}(\varphi) + c(\mathcal{F},\varphi) = p\tilde{f}, \tag{10}$$

where $\mathcal{F}=0$ *when* w *is solution of* (4), *and where*

$$q = (\mu - \tilde{b})/p \in C^{m-4,ee}_{\natural\natural}, \qquad c(\mathcal{F},\varphi) = \big\{p\Gamma\big(\mathcal{F}, \widehat{p^{-1}\varphi}\big)\big\}^{\sim},$$

and $\mathcal{G}$ *is a Q1D operator of order* $(1,\beta)$ *on* H^s *for* $0 \leqslant \beta + s \leqslant m-6$ *defined by*

$$\mathcal{G}(\varphi) = -\partial_y\big\{\mathcal{S}(q\varphi) + \mathcal{S}_q\varphi + \big\{\mathcal{H}\mathcal{S}_a\big(\widehat{p^{-1}\partial_t\varphi}\big)\big\}^{\sim}\big\}.$$

Moreover operators $\mathcal{G}$ *and* $c(\mathcal{F},\cdot)$ *satisfy the following estimates for* $s \geqslant 1$

$$\big\|G_{1,l-s}\mathcal{G}(\varphi)\big\|_{H^s} \leqslant c_l(M_6)\|w\|_{H^{l+6}}\|\varphi\|_{H^s}, \quad l \geqslant s,$$
$$\big\|c(\mathcal{F},\varphi)\big\|_{H^s} \leqslant c_s(M_7)\big\{\|\mathcal{F}\|_{H^{s+1}}\|\varphi\|_{H^3} + \|\mathcal{F}\|_{H^2}\big(\|w\|_{H^{s+6}}\|\varphi\|_{L^2} + \|\varphi\|_{H^{s+2}}\big)\big\},$$

where M_j *is defined by* $\|w\|_{H^j} \leqslant M_j$, *and for any real* α, β *we denote by* $G_{\alpha,\beta}$ *the operator on* $H^s_{\natural\natural}$ *defined by*

$$\mathfrak{F}(G_{\alpha,\beta}u)(\zeta) = \mathfrak{F}(u)\big(1+|\zeta_t|\big)^{-\alpha}\big(1+|\zeta_x|\big)^{\beta}, \quad \zeta \in \mathbb{Z}^2,$$

where $\mathfrak{F}$ *denotes the Fourier expansion.*

We are now in a position to use the method developed by Plotnikov and Toland in [7], which consists in making a suitable change of variables which transforms the linear equation (10) into a simpler one, where the main part has constant coefficients, these coefficients depending on the point $\underline{w}$ where we linearize. The remaining part is a Q1D operator which means that we can invert the full linear operator, provided Diophantine conditions, which depend on two coefficients in the equation, are fullfilled. More precisely, we can show the following

Theorem 2.2. *Consider the linearization $\partial_{\underline{w}}\underline{\mathcal{F}}(\underline{w},\varepsilon)u = f$ of (7) at $\underline{w} \in H^{m,ee}_{\natural\natural}$, $m \geqslant 12$. Then for $\|\underline{w}\|_{H^m}$ bounded and $0 \leqslant \varepsilon \leqslant \varepsilon_0$ small enough, there is a change of variable of the form*

$$\vartheta = \mathcal{P}^{-1}\{p\widetilde{L_{w'}u} \circ Q^{-1}\} = \mathcal{P}^{-1}\theta = \left[1 + \alpha_0 + \beta_0\mathcal{H} + (\alpha_1 + \beta_1\mathcal{H})\partial_\tau^{-1} + (\alpha_2 + \beta_2\mathcal{H})\partial_\tau^{-2}\right]\theta,$$
$$(\xi,\tau) = Q(y,t) = \big(y + d_0(y), t + e_0(y,t)\big),$$
$$\alpha_j, \beta_j \in C^{m-5-j}_{\natural\natural}, \quad d_0 \in C^{m-3,o}_{\natural}, \quad e_0 \in C^{m-4,eo}_{\natural\natural},$$

such that ϑ satisfies

$$\partial_{\tau\tau}\vartheta - (1+\delta)\mathcal{H}\partial_\xi\vartheta - \kappa\vartheta - (\lambda_0 + \lambda_1\mathcal{H})\partial_\tau^{-2}\vartheta - \mathcal{V}(\vartheta) - \sigma(\mathcal{F},\vartheta) = h, \tag{11}$$

where $h = \mathcal{P}^{-1}\{[p\tilde{f}(1+\dot{e}_0)^{-2}] \circ Q^{-1}\}$ and where δ and κ are constant coefficients (depending smoothly on $\underline{w}$), defined by

$$\frac{1}{1+\delta} = 2\pi\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} q(y,t)^{1/2}\,\mathrm{d}t\right)^{-2}\mathrm{d}y, \qquad \kappa = \frac{(1+\delta)^2}{16\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1+\dot{e}_0}{1+d_0'}\{\ddot{e}_0^2 q^{-2} - e_0'^2\}\,\mathrm{d}y\,\mathrm{d}t,$$

λ_0 and $\lambda_1 \in C^{m-9}_{\natural\natural}$, and $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3$ where $\mathcal{V}_1$ and $\mathcal{V}_2$ are Q1D operators in H^s of order $(\beta+2,\beta)$ and $(0,\beta-1)$ respectively when $0 \leqslant \beta + s \leqslant m - 11$, and $\mathcal{V}_3$ is such that $\partial_\tau^3\mathcal{V}_3$ is bounded in H^s, $0 \leqslant s \leqslant m - 11$. The operators $\mathcal{V}_j$ and σ satisfy the same type of technical estimates as $\mathcal{G}$ and c in Theorem 2.1. Moreover, using the decomposition (6) of w, where $w_\varepsilon^{(N)}$ is given by Lemma 1.2, there are smooth scalar functions δ_1 and κ_1 of ε such that

$$\begin{aligned}&\left|\delta - \frac{\varepsilon^2}{4} - \varepsilon^3\delta_1(\varepsilon)\right| \leqslant \varepsilon^p c(\underline{M_4})\|\underline{w}\|_{H^4},\\ &\left|\kappa - \frac{\varepsilon^4}{4}\left(\mathrm{card}(I) - \frac{1}{2}\right)\sum_{l\in I} l^2 - \varepsilon^5\kappa_1(\varepsilon)\right| \leqslant \varepsilon^{p+2} c(\underline{M_4})\|\underline{w}\|_{H^5},\end{aligned} \tag{12}$$

holds, and the coefficients λ_0, λ_1 are $\mathrm{O}(\varepsilon^2)$ while the Q1D operator $\mathcal{V}$ is $\mathrm{O}(\varepsilon)$.

3. Inversion of the approximate linearized operator

In this section we invert the linear operator $\mathcal{A} - \mathcal{B}$ where

$$\mathcal{A}\vartheta = \partial_{\tau\tau}\vartheta - (1+\delta)\mathcal{H}\partial_\xi\vartheta - \kappa\vartheta, \qquad \mathcal{B}\vartheta = (\lambda_0 + \lambda_1\mathcal{H})\partial_\tau^{-2}\vartheta + \mathcal{V}(\vartheta), \tag{13}$$

which is not exactly the linear operator occuring in (11), because we have omitted $\sigma(\mathcal{F},\vartheta)$. This term, which vanishes when w is solution of the problem, allows us to use the version of the Nash–Moser implicit function theorem in [7], to find an inverse of the approximate linear operator $\mathcal{A} - \mathcal{B}$, with suitable estimates (not given here).

Let us decompose ϑ and define a projection P_0 as follows

$$\vartheta = \Theta + \varepsilon\Upsilon, \quad \Theta = P_0\vartheta = \sum_{q\geqslant 1} u_{q^2}^{(q)}\cos q^2\xi\cos q\tau, \quad \Upsilon = \sum_{n\neq q^2,\ n+q>0} y_n^{(q)}\cos n\xi\cos q\tau. \tag{14}$$

Then define new operators

$$\mathcal{M}_\varepsilon = \varepsilon^{-2} P_0(\mathcal{A} - \mathcal{B}) P_0, \qquad \mathcal{E}_\varepsilon = \varepsilon^{-1} P_0 \mathcal{B}(\mathbb{I} - P_0), \qquad \mathcal{K}_\varepsilon = \varepsilon^{-2}(\mathbb{I} - P_0)\mathcal{B} P_0,$$
$$\Lambda_\varepsilon^{(0)} = (\mathbb{I} - P_0)\mathcal{A}(\mathbb{I} - P_0), \qquad \Lambda_\varepsilon^{(1)} = \varepsilon^{-1}(\mathbb{I} - P_0)\mathcal{B}(\mathbb{I} - P_0),$$

and the system to be inverted reads

$$\mathcal{M}_\varepsilon \Theta - \mathcal{E}_\varepsilon \Upsilon = \varepsilon^{-2} P_0 h, \qquad \big(\Lambda_\varepsilon^{(0)} - \varepsilon \Lambda_\varepsilon^{(1)}\big)\Upsilon - \varepsilon \mathcal{K}_\varepsilon \Theta = \varepsilon^{-1}(\mathbb{I} - P_0) h. \tag{15}$$

In the case *when the basic formal expansion is built in Lemma* 1.2 *with* $I = \{1\}$, we can show that the operator $\mathcal{M}_\varepsilon^{-1}\mathcal{E}_\varepsilon$ is bounded by a constant independent of ε, in any space $\mathcal{L}((\mathbb{I} - P_0)H_{\natural\natural}^{k,ee}, P_0 H_{\natural\natural}^{k,ee})$, and the operators $\mathcal{K}_\varepsilon \mathcal{M}_\varepsilon^{-1}\mathcal{E}_\varepsilon$ and $\Lambda_\varepsilon^{(1)}$ have the same Q1D properties as $\mathcal{B}$, uniformly in ε. Then the inversion reduces to the inversion of $\Lambda_\varepsilon^{(0)}$ where *a small divisor problem, depending on the coefficients* δ *and* κ, appears.

Remark 3. We know a priori from the above theorem that $\mathcal{B} = \mathrm{O}(\varepsilon)$ holds for any choice of subset $I \subset \mathbb{N}$. The above property for $\mathcal{M}_\varepsilon^{-1}\mathcal{E}_\varepsilon$ and $\mathcal{K}_\varepsilon \mathcal{M}_\varepsilon^{-1}\mathcal{E}_\varepsilon$ would be also true for other finite subsets I, if one can prove (heavy calculations would be required) that

$$\mathcal{B} P_0 = \mathrm{O}\big(\varepsilon^2\big), \quad \text{and} \quad \mathcal{M}_0^{-1} \text{ is regularizing of order 1 in } \xi, \text{ or 2 in } \tau$$

holds for such I (this is true for $I = \{1\}$).

Now fix $c > 0$ and look for solutions for which (δ, κ) satisfies the Diophantine condition

$$\big|q^2 - (1+\delta)p + \kappa\big| \geqslant c/q^2, \quad (p,q) \in \mathbb{N}^2, \ p \neq q^2. \tag{16}$$

It can be shown that the linear operator $\mathcal{A} - \mathcal{B}$ is then invertible with a loss of two derivatives in τ, or one derivative in ξ for ϑ with respect to h in (15), the norm of its inverse being of order $\mathrm{O}(\varepsilon^{-2})$. Returning to the approximate linear equation

$$\partial_{\underline{w}} \mathcal{F}(\underline{w}, \varepsilon) u - \Gamma(\mathcal{F}, L_{w'} u) = f$$

we find that the operator on the left has an inverse in $\mathcal{L}(H_{\natural\natural}^{s,ee}, H_{\natural\natural}^{s-2,ee})$ the norm of which we can estimate in terms of $\underline{w}$ and is $\mathrm{O}(\varepsilon^{-2})$ as $\varepsilon \to 0$. This factor ε^{-2} is then controlled along the iteration process by choosing N large enough in (6). Control of the Diophantine condition (16) can be achieved along the Newton iteration scheme of the Nash–Moser theorem, thanks to (12). This finally leads to Theorem 1.3 for a set $\mathcal{E}$ of "good εs" such that

$$(1/r)\,\mathrm{meas}\big\{\mathcal{E} \cap (0,r)\big\} \to 1 \quad \text{as } r \to 0. \tag{17}$$

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