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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

A stochastic differential equation from friction mechanics

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Abstract

The existence and uniqueness of solutions to multivalued stochastic differential equations of the second order on Riemannian manifolds are proved. The class of problem is motivated by rigid body and multibody dynamics with friction and an application to the spherical pendulum with friction is presented. ***To cite this article: F. Bernardin et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une équation différentielle stochastique en mécanique du frottement. On démontre l'existence et l'unicité de la solution d'un système d'équations stochastiques multivoques du deuxième ordre sur une variété riemannienne. L'étude de cette classe de problèmes est motivée par la dynamique du corps rigide, et plus généralement des problèmes multicorps. On présente une application au pendule sphérique avec frottement. ***Pour citer cet article : F. Bernardin et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Les problèmes de mécanique à nombre fini de degré de liberté avec frottement fournissent une large classe de systèmes différentiels multivoques naturellement posés sur des variétés riemanniennes (voir [2]).

Diverses applications technologiques, principalement le comportement de grandes structures bâties soumises à un tremblement de terre ou de robots dans un environnement incomplètement connu, conduisent à perturber ces systèmes par des termes stochastiques. Suivant une interprétation largement acceptée, le choix d'un terme stochastique est un choix de description de données incomplètes. Il conviendra alors de faire les hypothèses mathématiques qui permettent de faire des démonstrations et de comparer à l'expérience les résultats obtenus au moyen de simulations numériques.

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Nous nous plaçons donc sur une variété M riemannienne lisse, et nous considérons le système (2), dont nous allons expliquer les notations.

L'espace $\mathcal{E} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ est un espace de probabilité filtré vérifiant les conditions habituelles, et B_t est un mouvement brownien d-dimensionel standart défini sur $\mathcal{E}$ (voir [8] pour des définitions).

Le couple $(x(t), v(t))$ avec un abus de notation classique est un élément du fibré tangent. La notation δ désigne la différentielle de Stratonovich, $D_{\dot{x}(t)}v(t)$ est la dérivée covariante de Stratonovich au dessus de x de la semi-martingale continue v à l'instant t (se reporter à [9] ou [7] pour des précisions sur le transport parallèle stochastique).

L'opérateur A est maximal monotone dans chaque fibre du fibré tangent $\mathsf{T}M$, ce qui veut dire que si $(v, w)_x$ désigne le produit scalaire dans la fibre $\mathsf{T}_x M$, on doit avoir pour chaque $w_i \in A(t, x, v_i)$, $i = 1, 2$, la relation de monotonie (1), et il doit être maximal parmi les opérateurs vérifiant cette condition (voir [4]). Le vecteur $\nu(t, x, v)$ est tangent, c'est le drift de l'équation différentielle stochastique, et l'opérateur linéaire $\sigma(t, x, v)$ applique un espace fixe $\mathbb{R}^d$ dans l'espace tangent $\mathsf{T}_x M$: c'est la matrice de diffusion.

La condition initiale est une variable aléatoire η à valeurs dans le fibré tangent, dont le moment d'ordre 2 est fini, et le temps initial T_0 est un temps d'arrêt, à valeurs presque sûrement dans $[0, T]$.

Nous prouvons l'existence et l'unicité d'une solution $\mathcal{F}_t$-adaptée du problème (2) ; voir l'énoncé exact de la Proposition 2.1, ainsi que les hypothèses faites sur la variété, le drift, la diffusion et l'opérateur maximal monotone.

La démonstration repose sur trois idées : d'abord, il est possible de localiser, ce qui requiert une hypothèse technique toujours vérifiée si la variété M est compacte ou plus généralement de courbure bornée. Or les équations de la mécanique, écrites en coordonnées locales sans précaution, comportent des termes quadratiques d'origine géométrique difficiles à traiter en stochastique multivoque. Nous nous débarrassons de ces termes en transportant parallèlement la vitesse (stochastique) le long de la courbe des positions. Les images par ce procédé du drift et de la diffusion ne sont plus que localement lipschitziennes par rapport à la vitesse. Il faut donc les approcher par un drift et une diffusion lipschitziens en vitesse. Il reste à effectuer une étape de discrétisation partielle en temps rappelant la démonstration du théorème de Carathéodory pour l'existence de solutions d'équations différentielles ordinaires. On peut alors appliquer un théorème de Cépa [5] sur les systèmes stochastiques maximaux monotones du premier ordre.

La preuve fait appel à des domaines divers des mathématiques, ce qui contribue à sa technicité.

Nous avons montré que nos résultats permettent de traiter un problème simple : le pendule sphérique avec un frottement sec de Coulomb. C'est un exemple intéressant, parce que c'est l'exemple le plus simple pour lequel la variété de configuration M est non triviale.

1. Introduction

Motivated by problems from the mechanics of rigid bodies with frictional contact, we have studied a class of second order multivalued stochastic differential equations on a Riemannian manifold, for which we have obtained an existence and uniqueness result.

The choice of working on a manifold is the standard choice of mechanics with bilateral constraints, including for instance rigid body dynamics, multibody dynamics without loss of contact; it is indeed a natural choice in the realm of theoretical mechanics with a finite number of degrees of freedom (see [2]).

Modelling friction in robotics is very useful, since it can be assumed that no perfect contact exists; if, morevoer, we include in the exterior forces a stochastic term, we are naturally led to the class of models considered here.

Let us observe that this model is of substantial interest for applications. Indeed, if exterior forces are known only by their statistics, which is for instance the case of the forces created by an earthquake, and if we model a large man-made structure, such as a bridge, a building, an offshore platform or a tunnel, as an object with a finite number of degrees of freedom and friction at all contacts, we find the type of model considered here.

Our results could not be modified so as to treat an analogous stochastic multivalued system of first order. Indeed, the problem lies with the monotonicity, and we do not know how to define a monotone operator on an ordinary manifold for lack of an appropriate vector structure. However, we know how to express the monotonicity in the tangent bundle, provided that we use parallel transport in order to compare tangent vectors at different points.

The mathematical framework and the notations are described in Section 2, the idea of the proof is sketched in Section 3, Section 4 is devoted to the spherical pendulum; conclusions and perspectives are drawn in Section 5.

2. The mathematical setting

Let us describe now our notations and assumptions. The d-dimensional Riemannian manifold M is smooth, and it satisfies a technical hypothesis which is always true for compact manifolds or manifolds with bounded curvature tensor.

The tangent bundle is $\mathsf{T}M$ and the scalar product in the fibre at x is denoted by $(\,,\,)_x$. The corresponding norm is $|\,|_x$. Let ρ be a mapping of class C^1 from $[0,t]$ to M. The parallel transport along $\rho|_{[a,b]}$ can be defined, and it is an isometry $\tau(\rho|_{[a,b]})$ from $\mathsf{T}_{\rho(a)}$ to $\mathsf{T}_{\rho(b)}$ (see [1] for a proof). In particular, if x and y are close enough, there is a unique geodesic from x to y and the parallel transport along this geodesic is denoted by $\bar\tau(y,x)$, which maps T_xM into T_yM.

Our class of maximal monotone operators are maximal monotone in each fibre of the tangent bundle. More precisely, for each $t\in[0,T]$ and $x\in M$, $A(t,x)$ is a maximal monotone operator in T_xM^2, i.e., for each $v\in\mathsf{T}_xM$ the image of v by $A(t,x)$ is a closed convex subset $A(t,x,v)$ of T_xM. Given v_1 and v_2 in T_xM and $w_j\in A(t,x,v_j)$, the monotonicity assumption is

$$(w_2-w_1,v_2-v_1)_x\geqslant 0, \tag{1}$$

and $A(t,x)$ is maximal among all multivalued mapping possessing property (1) (see [4]).

The dependence of $A(t,x)$ on t and x will be described in assumption H3, and for this purpose, we need the projection of $w\in\mathsf{T}_xM$ onto $A(t,x,v)$: it is well defined provided that $A(t,x,v)$ is not empty and is written $A(t,x,v,w)$.

Let us define now the drift $\nu(t,x)$ and the diffusion matrix $\sigma(t,x)$. For all t and x, $\nu(t,x)$ is a map from T_xM to itself and $\sigma(t,x)$ is a continuously differentiable mapping from T_xM to the set of linear mappings from $\mathbb{R}^d$ to T_xM. The operator norm of a linear mapping from $\mathbb{R}^d$ equipped with a given fixed Euclidean metric to T_xM is denoted by $\|\,\|_x$.

It is convenient to write $\nu(t,x,v)$ and $\sigma(t,x,v)$ for the respective images of v by $\nu(t,x)$ and $\sigma(t,x)$.

The following assumptions essentially say that all the objects needeed in the proof are locally Lipschitz continuous with respect to position and time, and Lipschitz continuous with respect to velocity. They use a continuous, non negative function ℓ on $[0,T]$.

H0 For all bounded open subset $U\subset M$, there exists $C_0(U)$ such that for all close enough x_1 and x_2 in U, for all v_1 in $\mathsf{T}_{x_1}M$ and v_2 in $\mathsf{T}_{x_2}M$, for all t_1 and t_2 in $[0,T]$:

$$\left|\frac{\partial\|\sigma(t_1,x_1,v)\|_x^2}{\partial v}\bigg|_{v=v_1}-\frac{\partial\|\sigma(t_2,x_2,v)\|_x^2}{\partial v}\bigg|_{v=v_2}\right| \leqslant C_0(U)\big(\ell(t_1-t_2)+d_M(x_1,x_2)+\big|v_1-\bar\tau(x_1,x_2)v_2\big|_{x_1}\big).$$

H1 There exists a constant C_1 such that for all $t\in[0,T]$ and for all $(x,v)\in\mathsf{T}M$,

$$\big|\nu(t,x,v)\big|_x+\big\|\sigma(t,x,v)\big\|_x+\left|\frac{\partial\|\sigma(t,x,v)\|_x^2}{\partial v}\right|\leqslant C_1\big(1+d_M(x,x_0)+|v|_x\big).$$

Here x_0 is fixed in M and d_M is the geodesic distance.

H2 For all bounded open subset $U \subset M$, there exists $C_2(U)$ such that for all close enough x_1 and x_2 in U, for all v_1 in $\mathsf{T}_{x_1}M$ and v_2 in $\mathsf{T}_{x_2}M$, for all t_1 and t_2 in $[0,T]$:

$$\begin{aligned}&\left|\nu(t_1,x_1,v_1)-\bar\tau(x_1,x_2)\nu(t_2,x_2,v_2)\right|_{x_1}+\left\|\sigma(t_1,x_1,v_1)-\bar\tau(x_1,x_2)\sigma(t_2,x_2,v_2)\right\|_{x_1}\\&\quad\leqslant C_2(U)\big(\ell\big(|t_1-t_2|\big)+d_M(x_1,x_2)+\left|v_1-\bar\tau(x_1,x_2)v_2\right|_{x_1}\big).\end{aligned}$$

H3 For all bounded open subset $U \subset M$, there exists $C_3(U)$ such that for all close enough x_1 and x_2 in U, for all v_1 and w_1 in $\mathsf{T}_{x_1}M$ and v_2 and w_2 in $\mathsf{T}_{x_2}M$, for all t_1 and t_2 in $[0,T]$:

$$\begin{aligned}&\left|A(t_1,x_1,v_1,w_1)-\bar\tau(x_1,x_2)A(t_2,x_2,v_2,w_2)\right|_{x_1}\\&\quad\leqslant C_3(U)\big(\ell\big(|t_1-t_2|\big)+d_M(x_1,x_2)+\left|v_1-\bar\tau(x_1,x_2)v_2\right|_{x_1}+\left|w_1-\bar\tau(x_1,x_2)w_2\right|_{x_1}\big).\end{aligned}$$

We prove the following result, where we denote by δ the Stratonovich differential:

Proposition 2.1. *Let T be a strictly positive number, and let A, σ and ν satisfy assumptions* H0–H3. *Let $\mathcal{E}=(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},\mathbb{P})$ be a filtered probability space verifying the usual conditions. Let B_t be the standard $\mathbb{R}^d$-Brownian motion on $\mathcal{E}$, and let T_0 be an $\mathcal{F}_t$-stopping time with values in $[0,T]$ almost surely. Let η be an $\mathcal{F}_{T_0}$-measurable random variable which takes its values in $\mathsf{T}M$ and whose second order moment is finite. Denote by $D_{\dot x(t)}v(t)$ the Stratonovich covariant differential of the continuous semi-martingale v above x at time t. There exists a unique $\mathcal{F}_t$-adapted stochastic process $(x(t),v(t))$ with values in $\mathsf{T}M$ which satisfies the following multivalued stochastic differential system*:

$$\begin{cases}\delta x(t)=v(t)\,\mathrm{d}t,\\ D_{\dot x(t)}v(t)+A(t,x(t),v(t))\,\mathrm{d}t\ni\nu(t,x(t),v(t))\,\mathrm{d}t+\sigma(t,x(t),v(t))\delta B(t),\quad T_0\leqslant t\leqslant T,\\ (x(T_0),v(T_0))=\eta,\quad \textit{almost surely}.\end{cases}\tag{2}$$

3. The strategy of proof

The proof is extremely technical, but the main steps can be sketched. First, we localize the problem in small balls where all pairs of points can be connected by a unique geodesic. The technical assumption on the manifold enables us to provide coverings by balls of this type in a sufficiently uniform fashion. We have also to localize the initial data by a partition of unity.

Next, we transport the problem along the solution curve by parallel transport to the random initial position, which takes its values in a small ball U and then by parallel transport along geodesics to a given point x_0. This technique provides a system devoid of quadratic terms of geometrical origin. However, it does not look Markovian anymore, while initially, it is indeed Markovian.

We also use a change of function, in order to work with a vanishing initial time, instead of a random initial time. This provides a new Brownian motion, denoted by $W(t)$.

Then, we perform a semi-discretization which is somewhat reminiscent of the Carathéodory construction: we discretize the time variable by defining $t_i^p=2^{-p}iT$, and on the time interval $[t_i^p,t_{i+1}^p]$, we use the position data only up to time t_i^p, and we solve the stochastic differential equation:

$$\delta v^p(t)+\tilde A^n\big(t_i^p,x^p\big|_{[0,t_i^p]},v^p(t)\big)\,\mathrm{d}t\ni\tilde\nu^n\big(t_i^p,x^p\big|_{[0,t_i^p]},v^p(t)\big)\,\mathrm{d}t+\tilde\sigma^n\big(t_i^p,x^p\big|_{[0,t_i^p]},v^p(t)\big)\delta W(t),\tag{3}$$

with initial condition $v^p(t_i^p+0)=v^p(t_i^p-0)$. Here, the quantities decorated with a tilda denote the image of the quantities without the decoration, after all the transport and diffeomorphism operations; moreover, the upper index n refers to a regularization yielding Lipschitz continuous functions, whose Lipschitz constant depends on n.

The results of Cépa [5,6] apply and we obtain the existence of a solution of (3).

Then it remains to perform a number of passages to the limit, some of which are delicate, in particular, the passage to the limit with respect to n.

4. The spherical pendulum

The case of a spherical pendulum with Coulomb friction is covered by our model. The system consists of a material point with unit mass attached to a rigid stem of length 1 and of negligible mass, which is connected at its other end to a fixed point O by a connection authorizing only angular displacements. In our units, the gravity is 1. In addition to its weight, the pendulum is submitted, at its mobile end, to a dry friction of Coulomb's type (with coefficient μ) and a random force acting in the tangent plane of the sphere. The configuration manifold is the unit sphere $\mathbb{S}^2$. The initial conditions are $(x_0, v_0) \in \mathsf{T}\mathbb{S}^2$. In the present case, the Riemannian metric is the metric induced by the Euclidean metric of $\mathbb{R}^3$, and therefore, we will abuse notations and denote the scalar product in the tangent space at x to $\mathbb{S}^2$ without an index x, the same convention holding for the norm of tangent vectors.

Let H_1 and H_2 be two bounded and locally Lipschitz continuous sections of $\mathsf{T}\mathbb{S}^2$.

Let (B^1, B^2) be a two-dimensional Brownian motion; we write formally the stochastic part of the exterior forces as (in Itô form)

$$F_s(t,x)\,\mathrm{d}t = \mathrm{d}B^1(t)H_1(x) + \mathrm{d}B^2(t)H_2(x).$$

Thanks to the boundedness and the local Lipschitz continuity of H_1 and H_2, the linear mapping

$$\sigma(x)\big(v^1, v^2\big) = v^1 H_1(x) + v^2 H_2(x)$$

satisfies H1 and H2 of Section 2.

The normal components of the acceleration and the weight are respectively $a_N(x,v) = -\|v\|^2 x$ and $P_N(x) = p^n(x)x$, with $p^n(x) = -x_3$. Therefore, the sum of the reaction of the support and of the tension of the stem is

$$R(x,v) + F(x,v) = -\big(\|v\|^2 + p^n(x)\big)x.$$

The friction force is then given by the multivalued relation

$$F_r(x,v) \in \begin{cases} \{0\} & \text{if } p^n(x) \leqslant -\|v\|^2, \\ -\mu\big\{\big(p^n(x) + \|v\|^2\big)v/\|v\|\big\} & \text{if } p^n(x) > -\|v\|^2 \text{ and } v \neq 0, \\ \mu\big\{u \in \mathsf{T}_x\mathbb{S}^2, \|u\| \leqslant p^n(x)\big\} & \text{if } p^n(x) > 0 \text{ and } v = 0. \end{cases} \tag{4}$$

It can be proved that $-F_r$ is a monotone operator A in the fiber, i.e., for all $w \in -F_r(x,v)$ and $w' \in -F_r(x,v')$, the following inequality holds:

$$(w - w', v - v') \geqslant 0. \tag{5}$$

Let $P = (0,0,1)$ be the weight and let $\nu(x,v) = P_T(x) = P + x_3 x$, be its tangential component. Then, the equation of the dynamic applied to the pendulum takes the form of (2):

$$\begin{cases} \mathrm{d}x(t) = v(t)\,\mathrm{d}t, \\ D_{\dot{x}(t)}v(t) + A(x(t), v(t))\,\mathrm{d}t \ni \nu(x(t), v(t))\,\mathrm{d}t + \sigma(x(t))\,\mathrm{d}B(t), \quad 0 \leqslant t < T, \\ (x(0), v(0)) = (x_0, v_0). \end{cases}$$

Assumptions H0–H3 are verified, as can be checked in a straightforward manner. Therefore, Proposition 2.1 applies.

5. Conclusion and perspectives

We have thought of a different proof strategy, which would have always kept the Markovian character of the stochastic processes. We could have written the local problems directly in the local charts, keeping the quadratic terms of geometrical origin. In order to estimate them, we would have needed to estimate the change of velocity after parallel transport along a loop in M. This can be done provided that the curvature tensor is bounded, and the loop encloses a region of bounded 2-area. We intend to produce such a proof, but up to now, we have been held up by significant mathematical difficulties.

There remains a large number of theoretical and applied questions in the present area. The first and very natural question would be to find the Fokker–Planck equation associated to stochastic friction problems in mechanics. We are aware that this may be very difficult and probably not very applicable: we expect the equation to be defined only in high spatial dimension and therefore numerical simulations would be very difficult to perform.

The numerical simulation of the problem is also an interesting question, since there are no standard methods; of course, we expect that the Euler stochastic scheme, which is well known in the Euclidean case (see [3]), will run. But can we do better? Can we improve the efficiency of the simulations? What kind of precision do we get? These are all open questions.

The technological applications include robotics, and in particular robotics of bipeds. It is well known that a biped (or a quadruped) can walk only because there is friction – see what happens on a frozen lake in winter. Including a stochastic term is a natural idea, and we expect that if the models are well posed, the stochastic part should not be too much of a disturbance. However, the reader should observe here that our model does *not* include impacts. Impact dynamics with a stochastic term seems much harder than friction dynamics with a stochastic term.

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