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Asymptotic normality of the ET method – application to the ET test

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Abstract

We propose a procedure to test the adequacy of the tail of a given F_0 to extreme observations and to check that this tail provides reasonable extrapolations above the maximal observation. The test is based on the asymptotic distribution of the ET (Exponential Tail) estimate of extreme quantiles which is established in a companion Note. The asymptotic level and power of the test are studied for several classes of distributions. ***To cite this article: J. Diebolt et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Normalité asymptotique de la méthode ET – application au test ET. Nous proposons un test permettant de vérifier l'adéquation de la queue d'une fonction de répartition F_0 aux observations extrêmes et de contrôler si cette queue fournit des extrapolations raisonnables au-delà de l'observation maximale. Le test est basé sur la loi asymptotique de l'estimateur ET (Exponential Tail) des quantiles extrêmes qui est établie dans une Note jointe. Le niveau et la puissance asymptotiques du test sont étudiés pour plusieurs classes de lois. ***Pour citer cet article : J. Diebolt et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Le test ET est motivé par des questions issues du domaine de la fiabilité des structures, où les lois usuelles sont dans le domaine d'attraction de Gumbel, DA(Gumbel), et ont un point terminal infini. Etant donné un échantillon $X_1, \ldots, X_n$ de fonction de répartition F, il s'agit de vérifier si un modèle $F_0 \in$ DA(Gumbel) fournit une approximation acceptable de la queue de la loi autour ou au-delà de l'observation maximale. Cette question est importante par exemple dans le contexte de la fiabilité des structures où des événements de faible probabilité peuvent avoir de fortes conséquences telles que des défaillances critiques ou des charges extrêmes. Nous supposons

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que, pour des niveaux de significations raisonnables, les tests d'adéquation usuels n'ont pas rejeté l'hypothèse nulle $\mathcal{H}_0 : \{F = F_0\}$. De telles procédures testent essentiellement l'adéquation du modèle dans la partie centrale de l'échantillon. Or, les dangers d'extrapoler dans les queues de la distribution les résultats de tels tests sont bien connus et détaillés par exemple dans [7] et [8]. Le but du test ET est de vérifier l'adéquation de la queue d'une fonction de répartition F_0 aux observations extrêmes et de contrôler si cette queue fournit des extrapolations raisonnables au-delà de l'observation maximale. En conséquence, nous testons

$$\mathcal{H}_0 : \{F = F_0\} \quad \text{contre} \quad \mathcal{H}_1 : \{F = F_1\}$$

dans la queue supérieure de la loi. Le principe du test ET est de comparer deux estimateurs différents d'un quantile extrême x_{p_n} défini par $\overline{F}(x_{p_n}) = p_n$ avec $p_n < 1/n$. Le premier est l'estimateur sous $\mathcal{H}_0$,

$$x_{\mathrm{param},\,n} = \overline{F}_0^{-1}(p_n).$$

Le second est l'estimateur ET défini par (1.3) dans [6] :

$$\hat{x}_{\mathrm{ET},\,n} = u_n + \hat{\sigma}_n \ln\left(\frac{N_n}{np_n}\right).$$

La suite de seuils déterministes u_n est telle que $\overline{F}(u_n) = m_n/n$ avec $1 \leqslant m_n \leqslant n,\ m_n \to \infty,\ m_n/n \to 0$ quand $n \to \infty$. La suite N_n représente le nombre (aléatoire) d'observations au-delà de u_n et $\hat{\sigma}_n$ est la moyenne empirique des excès eux-mêmes définis à partir des $X_i > u_n$ par $Y_i = X_i - u_n$. Lorsque le quantile est estimé par la méthode ET, deux erreurs sont commises : une erreur d'estimation et une erreur d'approximation puisque la loi des excès est approchée par une loi exponentielle. Sous $\mathcal{H}_0$, en ajoutant un équivalent $d_{0,n}$ de l'erreur d'approximation $\delta_{0,n}$ (voir [6], Remarque 2.3) aux bornes de l'intervalle de confiance déduit de la loi asymptotique de $\hat{x}_{\mathrm{ET},\,n}$ (voir [6], Théorème 1) on obtient un intervalle de confiance approché pour le vrai quantile :

$$IC_{\alpha,\,n} = \left[\hat{x}_{\mathrm{ET},\,n} + d_{0,n} \pm \hat{\sigma}_n \ln(r_n) m_n^{-1/2} z_\alpha\right],$$

où $r_n = m_n/(np_n)$ et z_α est tel que $P(|\xi| > z_\alpha) = \alpha$ avec $\xi \sim N(0,1)$. Le test ET rejette $\mathcal{H}_0$ quand l'estimateur sous $\mathcal{H}_0$ n'appartient pas à cet intervalle : $x_{\mathrm{param},\,n} \notin IC_{\alpha,n}$. Il s'agit là d'une version simplifiée du test ET proposé dans [4]. La version complète teste

$$\mathcal{H}_0 : \{F = F_\theta\} \quad \text{contre} \quad \mathcal{H}_1 : \{F \neq F_\theta\}$$

lorsque θ est estimé (par maximum de vraisemblance par exemple). De plus, [4] propose également de remplacer l'approximation normale asymptotique par des simulations bootstrap de $\hat{x}_{\mathrm{ET},\,n} - x_{\mathrm{param},\,n}$. Nous montrons ici que, pour les classes de lois introduites dans [6], le niveau du test ET simplifié converge vers α et sa puissance vers 1 quand $n \to \infty$.

1. The ET test

The ET test is motivated by questions arising in the field of Structural Reliability, where the usual distributions are in the Gumbel's Maximum Domain of Attraction, DA(Gumbel), and have infinite endpoint. Given a sample $X_1, \ldots, X_n$ of cumulative distribution function F, we wish to check that a model $F_0 \in$ DA(Gumbel) provides an acceptable approximation to the tail of the distribution in the range near and above the maximal observation. This is important since in the context of Structural Reliability events with low probabilities can imply strong consequences as critical failures or extreme charges. We assume that at reasonable significance levels, usual goodness-of-fit tests have not rejected the null hypothesis $\mathcal{H}_0$ that $\{F = F_0\}$. Such procedures essentially test the adequacy of the model to the central range of the sample, that is to say the central part of the sample interval. The dangers of extrapolating in the tails from the results of such tests are detailed, e.g., in [7] and in [8]. The purpose of the ET test is to check the adequacy of the tail of a given F_0 to extreme observations and to check that this tail provides reasonable extrapolations above the maximal observation. Therefore, we wish to test

$$\mathcal{H}_0 : \{F = F_0\} \quad \text{against} \quad \mathcal{H}_1 : \{F = F_1\}$$

in the upper tail. The principle of the ET test is to compare two different estimates of some extreme quantile x_{p_n} defined by $\overline{F}(x_{p_n}) = p_n$ with $p_n < 1/n$. The first one is the estimate under $\mathcal{H}_0$,

$$x_{\text{param},\,n} = \overline{F}_0^{-1}(p_n).$$

The second one is the ET estimate defined by (1.3) in [6]:

$$\hat{x}_{\text{ET},\,n} = u_n + \hat{\sigma}_n \ln\left(\frac{N_n}{np_n}\right).$$

The sequence of deterministic thresholds u_n is such that $\overline{F}(u_n) = m_n/n$, with $1 \leqslant m_n \leqslant n,\ m_n \to \infty,\ m_n/n \to 0$ as $n \to \infty$. The sequence N_n represents the random number of observations above u_n and $\hat{\sigma}_n$ is the empirical mean of the excesses defined on the basis of the $X_i > u_n$'s by $Y_i = X_i - u_n$.

When we estimate the true quantile by its ET estimation, we make two kinds of errors: an estimation error and an approximation error, since we approximate the excess distribution with an exponential distribution. Under $\mathcal{H}_0$, adding an asymptotic equivalent $d_{0,n}$ of the approximation error $\delta_{0,n}$ (see [6], Remark 2.3) to the bounds of the confidence interval based on the asymptotic distribution of $\hat{x}_{\text{ET},\,n}$ (see [6], Theorem 1) yields an approximate confidence interval for the true quantile:

$$CI_{\alpha,\,n} = \left[\hat{x}_{\text{ET},\,n} + d_{0,n} \pm \hat{\sigma}_n \ln(r_n) m_n^{-1/2} z_\alpha\right],$$

with $r_n = m_n/(np_n)$ and z_α is such that $P(|\xi| > z_\alpha) = \alpha$ with $\xi \sim N(0,1)$. The ET goodness-of-fit test rejects $\mathcal{H}_0$ when the estimate under $\mathcal{H}_0$ does not lay within this confidence interval: $x_{\text{param},\,n} \notin CI_{\alpha,n}$. This is a simplified version of the ET test proposed in [4]. The full version aims at testing $\mathcal{H}_0 : \{F = F_\theta\}$ against $\mathcal{H}_1 : \{F \neq F_\theta\}$ where θ is estimated (e.g., by maximum likelihood).

2. Asymptotic level and power

The behaviour of the asymptotic level and power of the test is established for the classes of distributions already considered in [6]: $\mathcal{C} = \mathcal{C}_\theta^1 \cup \mathcal{C}^2 \cup \mathcal{C}_\theta^3$ with

$$\begin{aligned}
&\mathcal{C}_\theta^1 = \mathcal{SR}_{1/\theta}, \quad \theta > 0,\ \theta \neq 1,\\
&\mathcal{C}_1^1 = \mathcal{C}_{1,\infty}^1 \cup \mathcal{C}_{1,\tau}^1 = \{V \in \mathcal{SR}_1\colon\ V'' = 0\} \cup \left\{V \in \mathcal{SR}_1\colon\ |V''| \in \mathcal{SR}_{-1-\tau}\right\}, \quad \tau \geqslant 0,\\
&\mathcal{C}^2 = \{V \in \mathcal{SR}_0,\ V' \in \mathcal{SR}_{-1}\},\\
&\mathcal{C}_\theta^3 = \{V = \exp g,\ g \in \mathcal{SR}_\theta,\ 0 < \theta < 1\},
\end{aligned}$$

where $\mathcal{SR}_\theta$ is the subset of smooth regularly varying functions in $\mathcal{RV}_\theta$, [1]. For the sake of simplicity (see [3]), we assume that

$$m_n \text{ is the largest integer } \leqslant \text{cst}_1\, n^{1-p} (\ln n)^{-q} \quad \text{and} \quad p_n = \text{cst}_2\, n^{-p'} (\ln n)^{-q'} \tag{1}$$

for some positive constants cst_1 and cst_2, where $0 < p \leqslant 1$ and $q > 0$ when $p = 1$ (with no constraint on q when $0 < p < 1$), and $p' \geqslant 1$ and $q' \geqslant 0$ when $p' = 1$ (with no constraint on q' when $p' > 1$).

Introducing $V_0(x) = \overline{F}_0^{-1}(\mathrm{e}^{-x})$ and $V_1(x) = \overline{F}_1^{-1}(\mathrm{e}^{-x})$, we have the following two results:

Theorem 2.1. *Consider two sequences* (m_n) *and* (p_n) *verifying* (1). *In each of the following cases*:

(i) $V_0 \in \mathcal{C}_\theta^1 \cup \mathcal{C}^2,\ \theta \neq 1,\ p = p' = 1$ *and* $q < \min(2, q')$,

(ii) $V_0 \in \mathcal{C}^1_{1,\infty}$, $p < p'$ *or* $q < q'$,
(iii) $V_0 \in \mathcal{C}^1_{1,\tau}$, $p = p' = 1$ *and* $q < \min(2(1+\tau), q')$,
(iv) $V_0 \in \mathcal{C}^3_{\theta}$, $p = p' = 1$ *and* $q < \min(2(1-\theta), q')$,

the level of the ET test converges to α *as* $n \to \infty$.

Theorem 2.2. *Consider two sequences* (m_n) *and* (p_n) *verifying* (1). *In each of the following cases*:

(i) $V_0 \in \mathcal{C}^1_{\theta}$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^1_{\theta}$, $\theta \neq 1$, $p = p' = 1$ *and* $q < \min(2, q')$,
(ii) $V_0 \in \mathcal{C}^1_{1,\infty}$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^1_{1,\infty}$, $p < p'$ *or* $q < q'$,
(iii) $V_0 \in \mathcal{C}^1_{1,\tau}$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^1_{1,\tau}$, $p = p' = 1$ *and* $q < \min(2(1+\tau), q')$,
(iv) $V_0 \in \mathcal{C}^2$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^2$, $p = p' = 1$ *and* $q < \min(2, q')$,
(v) $V_0 \in \mathcal{C}^3_{\theta}$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^3_{\theta}$, $p = p' = 1$ *and* $q < \min(2(1-\theta), q')$,

the power of the ET test converges to 1 as $n \to \infty$.

Let us note that these theorems give no information on the speed of convergence. Nevertheless, Theorem 2.1 in [6] shows that $\hat{x}_{\mathrm{ET},\,n} - x_{\mathrm{param},\,n}$ converges to a Gaussian random variable at a logarithmic speed. Therefore, the above convergences should be very slow too. This problem has been illustrated on simulations in [4] and a bootstrap procedure has been proposed to overcome it.

3. Proofs

The study of the asymptotic level and power of the ET test requires further notations. For $k \in \{0, 1\}$ we introduce:

- The true quantile under $\mathcal{H}_k$: $x_{k,p_n} = \overline{F}_k^{-1}(p_n) = V_k(b_n)$ with $b_n = \ln(1/p_n)$;
- Its ET approximation under $\mathcal{H}_k$: $x_{\mathrm{ET},\,k,n} = u_{k,n} + \sigma_k(u_{k,n})\ln(r_n)$ where $u_{k,n} = \overline{F}_k^{-1}(n/m_n) = V_k(a_n)$ with $a_n = \ln(n/m_n)$, and $\sigma_k(u_{k,n}) = V_k'(V_k^{-1}(u_{k,n}))$;
- The corresponding error of approximation under $\mathcal{H}_k$: $\delta_{k,n} = x_{k,p_n} - x_{\mathrm{ET},\,k,n}$;
- Its first order approximation under $\mathcal{H}_k$ (see [6], Remark 2.3): $d_{k,n} = (\ln r_n)^2 V_k''(\varrho_n)/2$ with $\varrho_n \in [b_n, a_n]$;
- The auxiliary function $\Omega_k(t) = t V_k'(t)/V_k(t)$, $t > 0$.

We also define $\Omega(t) = V_0(t)/V_1(t)$, $t > 0$, and the following sequences

$$\omega_n = \sqrt{m_n}\,\frac{d_{0,n} - \delta_{0,n}}{\sigma_0(u_{0,n})\ln(r_n)}, \quad \phi_n = \sqrt{m_n}\,\frac{x_{\mathrm{ET},\,1,n} - x_{\mathrm{ET},\,0,n}}{\sigma_1(u_{1,n})\ln(r_n)} \quad \text{and} \quad \psi_n = \frac{\sigma_0(u_{0,n})}{\sigma_1(u_{1,n})}. \tag{2}$$

The next two lemmas describe the asymptotic behavior of these sequences which drive the asymptotic level and power of the ET test. Their proofs can be found in [5].

Lemma 3.1. *Under the conditions of Theorem* 2.1, $\omega_n \to 0$ *as* $n \to \infty$.

Lemma 3.2. *Consider two sequences* (m_n) *and* (p_n) *verifying* (1). *In each of the following cases*:

(i) $V_0 \in \mathcal{C}^1_{\theta}$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^1_{\theta}$,
(ii) $V_0 \in \mathcal{C}^2$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^2$,
(iii) $V_0 \in \mathcal{C}^3_{\theta}$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^3_{\theta}$,

we have $|\phi_n| \to \infty$ *and* $\psi_n = \mathrm{o}(\phi_n)$ *as* $n \to \infty$.

Proof of Theorem 2.1. From [6], Theorem 1, and under $\mathcal{H}_0$,

$$x_{\text{param},\,n} - \hat{x}_{\text{ET},\,n} = x_{\text{param},\,n} - x_{\text{ET},\,0,n} + \sigma_0(u_{0,n})\ln(r_n)m_n^{-1/2}\,\xi_n,$$

where $\xi_n \xrightarrow{d} \xi \sim N(0,1)$. Therefore,

$$P(x_{\text{param},\,n} \notin CI_{\alpha,n} \mid \mathcal{H}_0) = P(\xi_n \notin J_{\alpha,n}),$$

where $J_{\alpha,n} = [\omega_n \pm z_\alpha \hat{\sigma}_n/\sigma_0(u_{0,n})]$ and ω_n is defined by (2). Now, since $m_n \to \infty$ and $m_n/n \to 0$ with (1), under $\mathcal{H}_0$ we have $\hat{\sigma}_n/\sigma_0(u_{0,n}) \xrightarrow{P} 1$ as $n \to \infty$ [2]. We thus have

$$P\big(\hat{\sigma}_n/\sigma_0(u_{0,n}) > 1+\eta\big) \to 0 \quad \text{and} \quad P\big(\hat{\sigma}_n/\sigma_0(u_{0,n}) < 1-\eta\big) \to 0,$$

when $n \to \infty$ for all $\eta > 0$. Introducing $K_{\alpha,\,n} = [\omega_n \pm z_\alpha(1+\eta)]$, it follows that $\forall \eta > 0$,

$$P(J_{\alpha,n} \subset K_{\alpha,\,n}) = P\big(\hat{\sigma}_n/\sigma_0(u_{0,n}) \leqslant 1+\eta\big) = 1 - P\big(\hat{\sigma}_n/\sigma_0(u_{0,n}) > 1+\eta\big) \to 1. \tag{3}$$

From Lemma 3.1, $\omega_n \to 0$ as $n \to \infty$, and therefore there exists $N_0(\eta) \in \mathbb{N}$ such that $\forall n \geqslant N_0(\eta)$, $K_{\alpha,n} \subset [\pm z_\alpha(1+2\eta)]$ leading to

$$P(J_{\alpha,n} \subset K_{\alpha,\,n}) \leqslant P\big(J_{\alpha,n} \subset \big[\pm z_\alpha(1+2\eta)\big]\big). \tag{4}$$

As a consequence of (3) and (4), $\mathrm{P}(J_{\alpha,n} \subset [\pm z_\alpha(1+2\eta)]) \to 1$ as $n \to \infty$. This is equivalent to

$$\forall \nu_1 > 0,\ \exists N_1(\nu_1,\eta) \in \mathbb{N} \ \text{ such that } \forall n \geqslant N_1(\nu_1,\eta), \quad P\big(J_{\alpha,n} \not\subset \big[\pm z_\alpha(1+2\eta)\big]\big) < \nu_1.$$

It follows that $\forall n \geqslant N_1(\nu_1,\eta)$,

$$\begin{aligned} P(\xi_n \in J_{\alpha,n}) &= P\big(\{\xi_n \in J_{\alpha,n}\} \cap \big\{J_{\alpha,n} \subset \big[\pm z_\alpha(1+2\eta)\big]\big\}\big) + P\big(\{\xi_n \in J_{\alpha,n}\} \cap \big\{J_{\alpha,n} \not\subset \big[\pm z_\alpha(1+2\eta)\big]\big\}\big) \\ &\leqslant P\big(\xi_n \in \big[\pm z_\alpha(1+2\eta)\big]\big) + P\big(J_{\alpha,n} \not\subset \big[\pm z_\alpha(1+2\eta)\big]\big) \leqslant P\big(\xi_n \in \big[\pm z_\alpha(1+2\eta)\big]\big) + \nu_1. \end{aligned} \tag{5}$$

Moreover, for all $\nu_2 > 0$, there exists $C(\nu_2) > 0$ and $N_2(\nu_2)$ such that

$$\forall \eta \in \big]0,\, C(\nu_2)\big] \text{ and } \forall n \geqslant N_2(\nu_2), \quad \big|P\big(\xi_n \in \big[\pm z_\alpha(1+2\eta)\big]\big) - (1-\alpha)\big| < \nu_2,$$

which implies that

$$\forall \nu_2 > 0,\ \ \forall \eta \in \big]0,\, C(\nu_2)\big], \quad \limsup_{n\to\infty} P\big(\xi_n \in \big[\pm z_\alpha(1+2\eta)\big]\big) \leqslant 1-\alpha+\nu_2. \tag{6}$$

From (5) and (6), we obtain $\forall \nu_1, \nu_2 > 0$

$$\limsup_{n\to\infty} P(\xi_n \in J_{\alpha,n}) \leqslant 1-\alpha+\nu_1+\nu_2.$$

Similarly, it can be shown that $\forall \nu_1, \nu_2 > 0$,

$$\liminf_{n\to\infty} P(\xi_n \in J_{\alpha,n}) \geqslant 1-\alpha-\nu_1-\nu_2.$$

This entails: $\forall \nu_1, \nu_2 > 0$,

$$1-\alpha-\nu_1-\nu_2 \leqslant \liminf_{n\to\infty} P(\xi_n \in J_{\alpha,n}) \leqslant \limsup_{n\to\infty} P(\xi_n \in J_{\alpha,n}) \leqslant 1-\alpha+\nu_1+\nu_2.$$

Since ν_1 are ν_2 are arbitrarily small, both limits are equal and thus $P(\xi_n \in J_{\alpha,n}) \to 1-\alpha$ as $n \to \infty$. As a conclusion, $P(x_{\text{param},\,n} \notin CI_{\alpha,n} \mid \mathcal{H}_0) \to \alpha$ as $n \to \infty$. □

Proof of Theorem 2.2. From [6], Theorem 2.1, and under $\mathcal{H}_1$,

$$x_{\text{param},\,n} - \hat{x}_{\text{ET},\,n} = x_{\text{param},\,n} - x_{\text{ET},\,1,n} + \sigma_1(u_{1,n})\ln(r_n)m_n^{-1/2}\,\xi_n,$$

where $\xi_n \xrightarrow{d} \xi \sim N(0,1)$. Therefore, in view of (2),

$$P(x_{\text{param},\,n} \notin CI_{\alpha,n} \mid \mathcal{H}_1) = P(\xi_n \notin J'_{\alpha,n}), \tag{7}$$

where $J'_{\alpha,n} = [\omega_n \psi_n + \phi_n \pm z_\alpha \hat{\sigma}_n / \sigma_1(u_{1,n})]$. Now, since $m_n \to \infty$ and $m_n/n \to 0$ with (1), under $\mathcal{H}_1$, we have $\hat{\sigma}_n/\sigma_1(u_{1,n}) \xrightarrow{P} 1$ as $n \to \infty$ [2].

For $\eta > 0$, introducing $K'_{\alpha,\,n} = [\omega_n \psi_n + \phi_n \pm z_\alpha(1+\eta)]$, leads to $P(J'_{\alpha,n} \subset K'_{\alpha,\,n}) \to 1$ as $n \to \infty$. From Lemmas 3.1 and 3.2, $K'_{\alpha,\,n}$ can be rewritten as $K'_{\alpha,\,n} = [\phi_n(1+\varepsilon_n) \pm z_\alpha(1+\eta)]$, with $\varepsilon_n = \omega_n \psi_n / \phi_n \to 0$ as $n \to \infty$. It follows that $\forall \eta > 0,\ P(\xi_n \notin K'_{\alpha,\,n}) \to 1$ as $n \to \infty$, since $|\phi_n| \to \infty$. Remarking that

$$\begin{aligned} P(\xi_n \notin K'_{\alpha,\,n}) &= P\big(\{\xi_n \notin J'_{\alpha,n}\} \cap \{J'_{\alpha,n} \subset K'_{\alpha,\,n}\}\big) + P\big(\{\xi_n \notin J'_{\alpha,n}\} \cap \{J'_{\alpha,n} \not\subset K'_{\alpha,\,n}\}\big) \\ &\leqslant P(\xi_n \notin J'_{\alpha,n}) + P(J'_{\alpha,n} \not\subset K'_{\alpha,\,n}), \end{aligned}$$

it results that $1 \geqslant P(\xi_n \notin J'_{\alpha,n}) \geqslant P(\xi_n \notin K'_{\alpha,\,n}) - P(J'_{\alpha,n} \not\subset K'_{\alpha,\,n}) \to 1$ as $n \to \infty$. Therefore $P(\xi_n \notin J'_{\alpha,n}) \to 1$ as $n \to \infty$ and (7) concludes the proof. □

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