Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 337 (2003) 71–74
COMPTES RENDUS MATHEMATIQUE

Numerical Analysis

Mixed finite elements for incompressible magneto-hydrodynamics

Anna Schneebeli, Dominik Schötzau

Department of Mathematics, University of Basel, Rheinsprung 21, CH-4051 Switzerland

Received 4 April 2003; accepted after revision 16 May 2003

Presented by Philippe G. Ciarlet

Abstract

We present a new mixed finite element discretization for three-dimensional stationary incompressible magneto-hydrodynamics. The fluid variables are discretized by standard inf–sup stable velocity–pressure pairs and the magnetic variables by a mixed approach using Nédélec's elements of the first kind. The resulting method is shown to be quasi-optimally convergent. ***To cite this article: A. Schneebeli, D. Schötzau, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Méthode d'éléments finis mixtes pour la magnéto-hydrodynamique incompressible. Nous présentons une nouvelle méthode d'éléments finis mixtes pour les équations stationnaires tridimensionnelles de la magnéto-hydrodynamique incompressible. La partie fluide est discrétisée par des couples d'espaces standards vitesse–pression, stables selon la condition inf–sup, et la partie magnétique par une approche mixte utilisant les éléments de Nédélec de première espèce. Nous montrons que la méthode qui en résulte converge de façon quasi-optimale. ***Pour citer cet article : A. Schneebeli, D. Schötzau, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

1. Introduction

Incompressible magneto-hydrodynamics (MHD) describes the flow of a viscous, incompressible and electrically conducting fluid and arises in several engineering applications such as liquid metals in magnetic pumps or aluminum electrolysis. Over the last few years, several finite element approximations have been proposed for such problems that are based on nodal (i.e., H^1-conforming) finite elements for the magnetic field, combined with standard discretizations of the fluid variables. We mention here only [1,4–7] and the references therein. However, it has been known for some time that in non-convex polyhedra Ω of engineering practice, the magnetic field may have regularity below $H^1(\Omega)^3$ and that a nodal FEM discretization, albeit stable, can converge to a magnetic field that misses certain singular solution components induced by reentrant vertices or edges; see [2]. In this Note, we present an alternative mixed finite element approximation for incompressible MHD problems based on the Sobolev space $H(\mathrm{curl};\Omega)$. We use Nédélec's first family of elements for the discretization of the magnetic field, inf–sup stable velocity–pressure pairs for the hydrodynamic unknowns and standard elements for an additional Lagrange multiplier related to the divergence constraint on the magnetic field. The resulting method is shown to lead to quasi-optimal error bounds in general Lipschitz polyhedra.

E-mail addresses: schneeba@math.unibas.ch (A. Schneebeli), schotzau@math.unibas.ch (D. Schötzau).

1631-073X/03/$ – see front matter
doi:10.1016/S1631-073X(03)00256-5

For a Lipschitz domain $\Omega \subset \mathbb{R}^3$, we write $\|\cdot\|_s$ to denote the norm in the usual Sobolev space $H^s(\Omega)$, $s \geqslant 0$. The $L^2(\Omega)$-based inner product is $(\cdot,\cdot)$. We use the same notation for vector fields. We define $H_0^1(\Omega)$ as the subspace of functions in $H^1(\Omega)$ with zero trace on $\partial\Omega$. The space $H(\mathrm{curl};\Omega)$ is the space of vector fields $\vec{c} \in L^2(\Omega)^3$ with $\mathrm{curl}\,\vec{c} \in L^2(\Omega)^3$, endowed with the graph norm $\|\cdot\|_{\mathrm{curl}}$. $H_0(\mathrm{curl};\Omega)$ is the subspace of $H(\mathrm{curl};\Omega)$ of functions with zero tangential trace.

2. Mixed formulation of incompressible magneto-hydrodynamics

Let Ω be a bounded Lipschitz polyhedron in $\mathbb{R}^3$. For simplicity, we assume that Ω is simply-connected, and that its boundary $\partial\Omega$ is connected. The incompressible MHD problem we consider is to find the velocity field $\vec{u}$, the pressure p, the magnetic field $\vec{b}$, and the scalar function r satisfying

$$\begin{aligned} -R_s^{-1}\Delta\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \nabla p - S_c\,\mathrm{curl}\,\vec{b}\times\vec{b} &= \vec{f} \quad \text{in } \Omega,\\ R_m^{-1}S_c\,\mathrm{curl}(\mathrm{curl}\,\vec{b}) - S_c\,\mathrm{curl}(\vec{u}\times\vec{b}) - \nabla r &= \vec{g} \quad \text{in } \Omega,\\ \mathrm{div}\,\vec{u} = \mathrm{div}\,\vec{b} &= 0 \quad \text{in } \Omega. \end{aligned} \tag{1}$$

Here, R_s is the hydrodynamic Reynolds number, R_m the magnetic Reynolds number, S_c the coupling number, and $\vec{f}, \vec{g} \in L^2(\Omega)^3$ are given source terms. We complete the above system with the homogeneous boundary conditions $\vec{u} = \vec{0}$, $\vec{n}\times\vec{b} = \vec{0}$, and $r = \vec{0}$ on $\partial\Omega$, with $\vec{n}$ denoting the outward normal unit vector to $\partial\Omega$. Note that the scalar function r is the Lagrange multiplier associated to the constraint $\mathrm{div}\,\vec{b} = 0$. Its purpose is to render the formulation and its discretization stable; cf. [3]. By taking the divergence of the second equation in (1), we obtain $-\Delta r = \mathrm{div}\,\vec{g}$ in Ω, $r = 0$ on $\partial\Omega$. In particular, we have $r \equiv 0$ for a solenoidal source term $\vec{g}$.

By introducing the spaces $\vec{V} := H_0^1(\Omega)^3$, $Q := L^2(\Omega)/\mathbb{R}$, $\vec{C} := H_0(\mathrm{curl};\Omega)$, and $S := H_0^1(\Omega)$, the weak formulation of (1) reads: find $(\vec{u}, p, \vec{b}, r) \in \vec{V}\times Q\times\vec{C}\times S$ such that

$$\begin{aligned} a_s(\vec{u},\vec{v}) + c_0(\vec{u};\vec{u},\vec{v}) - c_1(\vec{b};\vec{v},\vec{b}) + b_s(p,\vec{v}) &= (\vec{f},\vec{v}),\\ a_m(\vec{b},\vec{c}) - c_2(\vec{b};\vec{u},\vec{c}) + b_m(r,\vec{c}) &= (\vec{g},\vec{c}),\\ b_s(q,\vec{u}) = b_m(s,\vec{b}) &= 0, \end{aligned} \tag{2}$$

for any $(\vec{v}, q, \vec{c}, s) \in \vec{V}\times Q\times\vec{C}\times S$. Here, we use the forms

$$\begin{aligned} &a_s(\vec{u},\vec{v}) := R_s^{-1}(\nabla\vec{u},\nabla\vec{v}), && a_m(\vec{b},\vec{c}) := R_m^{-1}S_c(\mathrm{curl}\,\vec{b},\mathrm{curl}\,\vec{c}),\\ &b_s(q,\vec{v}) := -(q,\mathrm{div}\,\vec{v}), && b_m(s,\vec{c}) := -(\nabla s,\vec{c}),\\ &c_1(\vec{d};\vec{v},\vec{b}) := S_c(\mathrm{curl}\,\vec{b}\times\vec{d},\vec{v}), && c_2(\vec{d};\vec{u},\vec{c}) := S_c(\vec{u}\times\vec{d},\mathrm{curl}\,\vec{c}),\\ &c_0(\vec{w};\vec{u},\vec{v}) := \tfrac{1}{2}\big((\vec{w}\cdot\nabla)\vec{u},\vec{v}\big) - \tfrac{1}{2}\big((\vec{w}\cdot\nabla)\vec{v},\vec{u}\big). \end{aligned}$$

The recent results in [9] show that the weak formulation in (2) is well-posed and that the following existence and uniqueness result holds.

Theorem 2.1. *For any $\vec{f}, \vec{g} \in L^2(\Omega)^3$, there exists at least one solution $(\vec{u}, p, \vec{b}, r)$ in $\vec{V}\times Q\times\vec{C}\times S$ of the mixed formulation in* (2)*. Moreover, there exists a constant C_Ω solely depending on Ω such that for small data with $(C_\Omega \max\{1, S_c\}[\|\vec{f}\|_0^2 + \|\vec{g}\|_0^2]^{\frac{1}{2}})/\min\{R_s^{-2}, R_m^{-2}S_c^2\} < 1$ the solution is unique.*

3. Finite element discretization

Let $\mathcal{T}_h$ be a regular and quasi-uniform partition of Ω into tetrahedra $\{K\}$. We denote by h_K the diameter of the element $K \in \mathcal{T}_h$, and set $h = \max_{K\in\mathcal{T}_h} h_K$.

To discretize the Navier–Stokes operator, we use standard finite element pairs $\vec{V}_h \subset \vec{V}$ and $Q_h \subset Q$, which are based on the mesh $\mathcal{T}_h$ and are assumed to be inf–sup stable independently of the mesh-size. We further assume the following standard approximation property to hold:

$$\inf_{\vec{v}\in\vec{V}_h} \|\vec{u}-\vec{v}\|_1 + \inf_{q\in Q_h} \|p-q\|_0 \leqslant Ch^{\min\{s,k\}}\big[\|\vec{u}\|_{s+1} + \|p\|_s\big] \tag{3}$$

for $(\vec{u}, p) \in H^{s+1}(\Omega)^3 \times H^s(\Omega)$, $s > \frac{1}{2}$, and an approximation order $k \geqslant 1$.

For the Maxwell operator, we use Nédélec's first family of spaces [8] combined with a standard H^1-conforming space. To this end, let $\mathcal{P}_k(K)$ be the space of polynomials of total degree $k \geqslant 0$ on K and $\widetilde{\mathcal{P}}_k(K)$ the space of homogeneous polynomials of degree k on K. The space $\mathcal{D}_k(K)$ denotes the polynomials $\vec{p}$ in $\widetilde{\mathcal{P}}_k(K)^3$ that satisfy $\vec{p}(\vec{x})\cdot\vec{x} = 0$ on K. For $k \geqslant 1$, we then set

$$\begin{aligned} \vec{C}_h &= \big\{\vec{c} \in \vec{C} | \vec{u}|_K \in \mathcal{P}_{k-1}(K)^3 \oplus \mathcal{D}_k(K),\ K \in \mathcal{T}_h\big\}, \\ S_h &= \big\{s \in S | s|_K \in \mathcal{P}_k(K),\ K \in \mathcal{T}_h\big\}. \end{aligned}$$

The finite element approximation of (2) is: find $(\vec{u}_h, p_h, \vec{b}_h, r_h) \in \vec{V}_h \times Q_h \times \vec{C}_h \times S_h$ such that

$$\begin{aligned} a_s(\vec{u}_h, \vec{v}) + c_0(\vec{u}_h; \vec{u}_h, \vec{v}) - c_1(\vec{b}_h; \vec{v}, \vec{b}_h) + b_s(p_h, \vec{v}) &= (\vec{f}, \vec{v}), \\ a_m(\vec{b}_h, \vec{c}) - c_2(\vec{b}_h; \vec{u}_h, \vec{c}) + b_m(r_h, \vec{c}) &= (\vec{g}, \vec{c}), \\ b_s(q, \vec{u}_h) = b_m(s, \vec{b}_h) &= 0, \end{aligned} \tag{4}$$

for any $(\vec{v}, q, \vec{c}, s) \in \vec{V}_h \times Q_h \times \vec{C}_h \times S_h$.

Using discrete Helmholtz decompositions, it can be easily seen that we have $r_h \equiv 0$ for a solenoidal source term $\vec{g}$. Furthermore, a discrete version of Theorem 2.1 has been established in [9].

4. Error analysis

The mixed method (4) leads to quasi-optimal error bounds.

Theorem 4.1. *Assume that* $(C_\Omega \max\{1, S_c\}[\|\vec{f}\|_0^2 + \|\vec{g}\|_0^2]^{1/2})/\min\{R_s^{-2}, R_m^{-2}S_c^2\} < \frac{1}{2}$. *Then we have the error bounds*

$$\begin{aligned} &\|\vec{u}-\vec{u}_h\|_1 + \|p-p_h\|_0 + \|\vec{c}-\vec{c}_h\|_{\mathrm{curl}} + \|r-r_h\|_1 \\ &\quad\leqslant C\Big[\inf_{\vec{v}\in\vec{V}_h} \|\vec{u}-\vec{v}\|_1 + \inf_{\vec{q}\in Q_h} \|p-q\|_0 + \inf_{\vec{c}\in\vec{C}_h} \|\vec{b}-\vec{c}\|_{\mathrm{curl}} + \inf_{s\in S_h} \|r-s\|_1\Big], \end{aligned}$$

with a constant $C > 0$ that is independent of the mesh-size.

Using Theorem 4.1, assumption (3) and standard approximation results for the spaces $\vec{C}_h$ and S_h yields the following convergence rates, see [9]. Let $(\vec{u}, p) \in H^{1+s}(\Omega)^3 \times H^s(\Omega)$, $r \in H^{1+s}(\Omega)$, $\vec{b} \in H^s(\Omega)^3$, $\mathrm{curl}\,\vec{b} \in H^s(\Omega)^3$, for a regularity exponent $s > \frac{1}{2}$. Then we have

$$\begin{aligned} &\|\vec{u}-\vec{u}_h\|_1 + \|p-p_h\|_0 + \|\vec{c}-\vec{c}_h\|_{\mathrm{curl}} + \|r-r_h\|_1 \\ &\quad\leqslant Ch^{\min\{s,k\}}\big[\|\vec{u}\|_{s+1} + \|p\|_s + \|\vec{b}\|_s + \|\mathrm{curl}\,\vec{b}\|_s + \|r\|_{s+1}\big]. \end{aligned}$$

5. Numerical results

We present numerical results for the following linearized and two-dimensional variant of (1):

$$-\Delta\vec{u} + \nabla p - \mathrm{curl}\,\vec{b}\times\vec{d} = \vec{f}, \quad \mathrm{curl}\,\mathrm{curl}\,\vec{b} - \mathrm{curl}(\vec{u}\times\vec{d}) - \nabla r = \vec{g}, \quad \mathrm{div}\,\vec{u} = \mathrm{div}\,\vec{b} = 0, \quad \text{in } \Omega,$$

Table 1
Energy errors and convergence rates for singular MHD solution

Mesh	Total dofs	H^1-error in $\vec{u}$		L^2-error in p		curl-error in $\vec{b}$		H^1-error in r
1	195	1.55×10^0	–	2.11×10^0	–	8.74×10^{-1}	–	1.17×10^{-9}
2	675	1.13×10^0	0.46	1.38×10^0	0.61	5.91×10^{-1}	0.57	1.95×10^{-9}
3	2499	8.03×10^{-1}	0.50	8.97×10^{-1}	0.63	4.00×10^{-1}	0.56	3.36×10^{-9}
4	9603	5.59×10^{-1}	0.52	5.95×10^{-1}	0.59	2.72×10^{-1}	0.55	2.01×10^{-9}
5	37635	3.86×10^{-1}	0.53	4.02×10^{-1}	0.57	1.86×10^{-1}	0.55	1.98×10^{-9}

where Ω is the L-shaped polygon $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$ and $\vec{d}$ the prescribed magnetic field $\vec{d} = (-1,1)$. We further choose $\vec{f}$, $\vec{g}$, and the boundary conditions so that the solution to the above problem is given by the strongest corner singularity for the underlying elliptic operator. The corresponding hydrodynamic variables $\vec{u}$ and p are then

$$\vec{u}(\vec{x}) = \begin{bmatrix} \rho^\lambda((1+\lambda)\sin(\phi)\psi(\phi) + \cos(\phi)\psi'(\phi)) \\ \rho^\lambda(-(1+\lambda)\cos(\phi)\psi(\phi) + \sin(\phi)\psi'(\phi)) \end{bmatrix}, \quad p(\vec{x}) = -\rho^{\lambda-1}\big((1+\lambda)^2\psi'(\phi) + \psi'''(\phi)\big)/(1-\lambda),$$

with $\psi(\phi) = \sin((1+\lambda)\phi)\cos(\lambda w)/(1+\lambda) - \cos((1+\lambda)\phi) - \sin((1-\lambda)\phi)\cos(\lambda w)/(1-\lambda) + \cos((1-\lambda)\phi)$, and with $\lambda \approx 0.5445$ and (ρ,ϕ) denoting the polar coordinates of $\vec{x} = (x_1, x_2)$. The pair $(\vec{b}, r)$ is given by $\vec{b}(\vec{x}) = \nabla(\rho^{2/3}\sin(2/3\phi))$ and $r(\vec{x}) \equiv 0$. We point out that the magnetic field $\vec{b}$ does not belong to $H^1(\Omega)^2$ and thus cannot be correctly captured by nodal elements; see [2].

The finite element approximations to this MHD solution are computed on a sequence of successively refined square meshes $\{\mathcal{T}_i\}_{i \geqslant 1}$, employing the general purpose finite element library `deal.II`; see [10]. The mesh-size in the mesh $\mathcal{T}_i$ is proportional to 2^{-i}. We use lowest order two-dimensional Nédélec's elements for the field $\vec{b}$, corresponding to rotated Raviart–Thomas elements, bilinear elements for r and inf–sup stable $Q_2^2 - Q_0$ elements for $(\vec{u}, p)$. In Table 1, we show the errors and numerical convergence rates that are obtained for each of the solution components. The numbers clearly show convergence in accordance with the theoretical results in Section 4. Note that the H^1-error for r vanishes within the accuracy of 10^{-8} that was used to iteratively solve the resulting linear systems. This test demonstrates the ability of our mixed method to resolve highly singular solutions whose magnetic components have regularity below $H^1(\Omega)$.

References

[1] F. Armero, J.C. Simo, Long-term dissipativity of time-stepping algorithms for an abstract evolution equation with applications to the incompressible MHD and Navier–Stokes equations, Comput. Methods Appl. Mech. Engrg. 131 (1996) 41–90.

[2] M. Costabel, M. Dauge, Weighted regularization of Maxwell equations in polyhedral domains, Numer. Math. 93 (2002) 239–277.

[3] L. Demkowicz, L. Vardapetyan, Modeling of electromagnetic absorption/scattering problems using *hp*-adaptive finite elements, Comput. Methods Appl. Mech. Engrg. 152 (1998) 103–124.

[4] J.-F. Gerbeau, A stabilized finite element method for the incompressible magnetohydrodynamic equations, Numer. Math. 87 (2000) 83–111.

[5] J.-L. Guermond, P. Minev, Mixed finite element approximation of an MHD problem involving conducting and insulating regions: the 2D case, Math. Model. Numer. Anal. 36 (2002) 517–536.

[6] J.-L. Guermond, P. Minev, Mixed finite element approximation of an MHD problem involving conducting and insulating regions: the 3D case, Numer. Methods Partial Differential Equations, to appear.

[7] M.D. Gunzburger, A.J. Meir, J.S. Peterson, On the existence and uniqueness and finite element approximation of solutions of the equations of stationary incompressible magnetohydrodynamics, Math. Comp. 56 (1991) 523–563.

[8] J.C. Nédélec, Mixed finite elements in $\mathbb{R}^3$, Numer. Math. 35 (1980) 315–341.

[9] D. Schötzau, Mixed finite element methods for stationary incompressible magneto-hydrodynamics, Tech. report 2003-03, Department of Mathematics, University of Basel.

[10] R. Hartmann, W. Bangerth, G. Kanschat, `deal.II` differential equations analysis library, technical reference, IWR, Universität Heidelberg, http://www.dealii.org.