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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Remarks on a Hardy–Sobolev inequality

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Abstract

We compute the optimal constant for a generalized Hardy–Sobolev inequality, and using the product of two symmetrizations we present an elementary proof of the symmetries of some optimal functions. This inequality was motivated by a nonlinear elliptic equation arising in astrophysics. ***To cite this article: S. Secchi et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Remarques sur une inégalité de Hardy–Sobolev. Nous calculons la meilleure constante dans une inégalité de Hardy–Sobolev généralisée, et en utilisant le produit de deux symétrisations, nous montrons de manière élémentaire la symétrie de certaines fonctions optimales. Cette inégalité est motivée par une équation elliptique non-linéaire en astrophysique. ***Pour citer cet article : S. Secchi et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Nous déterminons la constante optimale C dans l'inégalité de type Hardy–Sobolev

$$\int_{\mathbb{R}^N} \frac{|u(x)|^q}{|y|^\beta}\,\mathrm{d}x \leqslant C \int_{\mathbb{R}^N} |\nabla u|^p\,\mathrm{d}x, \tag{1}$$

où $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. Celle-ci est donnée par la valeur

$$C = \frac{p^p}{(k+p)^p}.$$

Nous considérons également la symétrie de certaines fonctions optimales. En utilisant le produit de deux symétrisations, nous prouvons qu'elles ne dépendent que de $(|y|, |z|)$.

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1. Introduction

The Hardy–Sobolev inequality

$$\int_{\mathbb{R}^N} \frac{|u(x)|^q}{|y|^\beta}\,dx \leqslant C \int_{\mathbb{R}^N} |\nabla u|^p\,dx, \tag{2}$$

where $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ was studied by Badiale and Tarantello in [1].

Our aim is to solve two open problems contained in [1]. First we compute the optimal value of the constant C in Eq. (2) in the case of Hardy's inequality, namely $p = q = \beta$. In fact we prove a more general inequality with optimal constant in Section 2. Second, in Section 3, we consider the symmetry of the optimal functions. Using the "product" of two symmetrizations, we prove that some optimal functions depend only on $(|y|, |z|)$.

1.1. Notation

- When we write an integral like $\int_{\mathbb{R}^N} |u|$, we mean that the integral is taken with respect to the Lebesgue measure on $\mathbb{R}^N$.
- $\mathcal{D}(\mathbb{R}^N)$ denotes the space of test functions, namely
  $$\mathcal{D}(\mathbb{R}^N) = \{u \in C^\infty(\mathbb{R}^N) \mid \operatorname{supp} u \text{ is compact}\}.$$
- $D^{1,p}(\mathbb{R}^N)$ is the closure of $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm
  $$\|u\|_{D^{1,p}} = \left(\int_{\mathbb{R}^N} |\nabla u|^p\right)^{1/p}.$$
- $W^{1,p}(\mathbb{R}^N)$ is the usual Sobolev space, namely the closure of $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm
  $$\|u\|_{W^{1,p}} = \left(\int_{\mathbb{R}^N} |\nabla u|^p + \int_{\mathbb{R}^N} |u|^p\right)^{1/p}.$$

2. Generalized Hardy inequality

If $1 \leqslant k \leqslant N$, we will write a generic point $x \in \mathbb{R}^N$ as $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$.

Theorem 2.1. *Let $1 < p < \infty$ and $\alpha + k > 0$. Then, for each $u \in \mathcal{D}(\mathbb{R}^N)$ the following inequality holds*:

$$\int_{\mathbb{R}^N} \left|u(x)\right|^p |y|^\alpha\,dx \leqslant \frac{p^p}{(\alpha + k)^p} \int_{\mathbb{R}^N} \left|\nabla u(x)\right|^p |y|^{\alpha+p}\,dx. \tag{3}$$

Moreover, the constant $p^p/(\alpha + k)^p$ is optimal.

Lemma 2.2. *Inequality* (3) *holds when $k = N$.*

Proof. See [3,6] and also [5], where many generalizations are proved. □

Lemma 2.3. *When $k = N$ the constant $p^p/(\alpha + N)^p$ in inequality* (3) *is optimal.*

Proof. Consider the family of functions

$$u_\varepsilon(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ |x|^{-(\alpha+N)/p-\varepsilon} & \text{if } |x| > 1 \end{cases}$$

and pass to the limit as $\varepsilon \to 0$. □

Lemma 2.4. *Inequality* (3) *holds for any* $1 \leqslant k \leqslant N$.

Proof. For every $u \in \mathcal{D}(\mathbb{R}^N)$ we have by Lemma 2.2

$$\int_{\mathbb{R}^N} |u(x)|^p |y|^\alpha \,\mathrm{d}x = \int_{\mathbb{R}^{N-k}} \mathrm{d}z \int_{\mathbb{R}^k} |u(x)|^p |y|^\alpha \,\mathrm{d}y \leqslant \frac{p^p}{(\alpha+k)^p} \int_{\mathbb{R}^{N-k}} \mathrm{d}z \int_{\mathbb{R}^k} |\nabla_y u(x)|^p |y|^{\alpha+p} \,\mathrm{d}y$$
$$\leqslant \frac{p^p}{(\alpha+k)^p} \int_{\mathbb{R}^N} |\nabla u(x)|^p |y|^{\alpha+p} \,\mathrm{d}y. \qquad \square$$

Lemma 2.5. *The constant* $p^p/(\alpha+k)^p$ *in inequality* (3) *is optimal.*

Proof. Let us choose $u : (y,z) \mapsto v(y)w(z)$ with $v \in \mathcal{D}(\mathbb{R}^k)$ and $w \in \mathcal{D}(\mathbb{R}^{N-k})$. It is clear that

$$\int_{\mathbb{R}^N} |\nabla u(x)|^p |y|^{\alpha+p} \,\mathrm{d}x = \int_{\mathbb{R}^N} \left(|\nabla v(y)|^2 w(z)^2 + |\nabla w(z)|^2 v(y)^2\right)^{p/2} |y|^{\alpha+p} \,\mathrm{d}x,$$

and

$$\int_{\mathbb{R}^N} |u(x)|^p |y|^\alpha \,\mathrm{d}x = \int_{\mathbb{R}^k} |v(y)|^p |y|^\alpha \,\mathrm{d}y \int_{\mathbb{R}^{N-k}} |w(z)|^p \,\mathrm{d}z.$$

If we consider the convex function

$$F : [0,+\infty) \times [0,+\infty) \to [0,+\infty), \quad (s,t) \mapsto \left(s^2+t^2\right)^{p/2},$$

we get that

$$\left(s^2+t^2\right)^{p/2} \leqslant (1-\lambda)^{1-p} s^p + \lambda^{1-p} t^p, \tag{4}$$

for all $s,t \geqslant 0$ and $0 < \lambda < 1$. Hence we obtain, for $0 < \lambda < 1$,

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^p |y|^{\alpha+p} \,\mathrm{d}x}{\int_{\mathbb{R}^N} |u|^p |y|^\alpha \,\mathrm{d}x} \leqslant (1-\lambda)^p \frac{\int_{\mathbb{R}^k} |\nabla v|^p |y|^{\alpha+p} \,\mathrm{d}y}{\int_{\mathbb{R}^k} |v|^p |y|^\alpha} + \lambda^p \frac{\int_{\mathbb{R}^{N-k}} |\nabla w|^p \,\mathrm{d}z}{\int_{\mathbb{R}^{N-k}} |w|^p \,\mathrm{d}z} \frac{\int_{\mathbb{R}^k} |v|^p |y|^{\alpha+p} \,\mathrm{d}y}{\int_{\mathbb{R}^k} |v|^p |y|^\alpha \,\mathrm{d}y}.$$

Since

$$\inf_{\substack{w \in \mathcal{D}(\mathbb{R}^{N-k}) \\ w \neq 0}} \frac{\int_{\mathbb{R}^{N-k}} |\nabla w|^p \,\mathrm{d}z}{\int_{\mathbb{R}^{N-k}} |w|^p \,\mathrm{d}z} = 0,$$

we obtain, for $0 < \lambda < 1$,

$$\inf_{\substack{u \in \mathcal{D}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p |y|^{\alpha+p} \,\mathrm{d}x}{\int_{\mathbb{R}^N} |u|^p |y|^\alpha \,\mathrm{d}x} \leqslant (1-\lambda)^p \inf_{\substack{v \in \mathcal{D}(\mathbb{R}^k) \\ v \neq 0}} \frac{\int_{\mathbb{R}^k} |\nabla v|^p |y|^{\alpha+p} \,\mathrm{d}y}{\int_{\mathbb{R}^k} |v|^p |y|^\alpha}.$$

By letting $\lambda \to 0$, we deduce from Lemma 2 that

$$\inf_{\substack{u\in\mathcal{D}(\mathbb{R}^N)\\ u\neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p |y|^{\alpha+p}\,\mathrm{d}x}{\int_{\mathbb{R}^N} |u|^p |y|^{\alpha}\,\mathrm{d}x} \leqslant \frac{(\alpha+k)^p}{p^p}.$$

The conclusion then follows from Lemma 2.4. □

Remark 1. When $k = N$ and $\alpha = -p$, inequality (2) is the classical Hardy inequality (see [3] or [6]).

Remark 2. When $2 \leqslant k \leqslant N$ and $\alpha = -p$, inequality (2) was conjectured by Badiale and Tarantello in [1].

Remark 3. This method is applicable in many other problems. A simple example is that for any open subset $\Omega \subset \mathbb{R}^M$,

$$\inf_{\substack{u\in\mathcal{D}(\Omega\times\mathbb{R}^N)\\ \int_{\Omega\times\mathbb{R}^N} |u|^p=1}} \int_{\Omega\times\mathbb{R}^N} |\nabla u|^p = \inf_{\substack{v\in\mathcal{D}(\Omega)\\ \int_{\Omega} |v|^p=1}} \int_{\Omega} |\nabla v|^p.$$

3. Cylindrical symmetry

In this section, we consider the minimization problem

$$S = S(N,p,k,\beta) = \inf\left\{ \int_{\mathbb{R}^N} |\nabla u|^p \mid u \in D^{1,p}\left(\mathbb{R}^N\right) \text{ and } \int_{\mathbb{R}^N} \frac{|u|^q}{|y|^\beta} = 1 \right\}, \tag{5}$$

where

(H) $0 \leqslant \beta < k,\ \beta \leqslant p,\ q = q(N,p,\beta) = \frac{p(N-\beta)}{N-p}$.

Let $u \in L^1(\mathbb{R}^N)$ be a non-negative function. Let us denote by $u^\star(\cdot, z)$ the Schwarz symmetrization of $u(\cdot, z)$ and by $u^{\star\star}(y,\cdot)$ the Schwarz symmetrization of $u^\star(y,\cdot)$. It is clear that $u^{\star\star}$ depends only on $(|y|,|z|)$. Let us define

$$D^{1,p}_{\star\star}\left(\mathbb{R}^N\right) = \left\{u \in D^{1,p}\left(\mathbb{R}^N\right) \mid u = u^{\star\star}\right\}$$

and

$$S^{\star\star} = S^{\star\star}(N,p,k,\beta) = \inf\left\{ \int_{\mathbb{R}^N} |\nabla u|^p \mid u \in D^{1,p}_{\star\star}\left(\mathbb{R}^N\right) \text{ and } \int_{\mathbb{R}^N} \frac{|u|^q}{|y|^\beta} = 1 \right\}. \tag{6}$$

Theorem 3.1. *Under assumption* (H), $S = S^{\star\star}$.

Since it is clear that $S \leqslant S^{\star\star}$, Theorem 3.1 follows from a density argument and the next lemma.

Lemma 3.2. *Under assumption* (H), *for each* $u \in \mathcal{D}(\mathbb{R}^N)$, *u non-negative, then*

$$\int_{\mathbb{R}^N} \left|\nabla u^{\star\star}(x)\right|^p \leqslant \int_{\mathbb{R}^N} \left|\nabla u(x)\right|^p, \qquad \int_{\mathbb{R}^N} \frac{|u^{\star\star}(x)|^q}{|y|^\beta} \geqslant \int_{\mathbb{R}^N} \frac{|u(x)|^q}{|y|^\beta}.$$

Proof. (a) By the Polya–Szegö inequality for Steiner symmetrization (see [2], Theorem 8.2), $u^\star$ and $u^{\star\star}$ belong to $W^{1,p}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \left|\nabla u^{\star\star}\right|^p \leqslant \int_{\mathbb{R}^N} \left|\nabla u^{\star}\right|^p \leqslant \int_{\mathbb{R}^N} |\nabla u|^p.$$

(b) Let $R > 0$ be such that

$$\operatorname{supp} u \subset B(0, R) \times \mathbb{R}^{N-k},$$

and define

$$v(y, z) = \begin{cases} \dfrac{1}{|y|^\beta} & \text{if } |z| \leqslant R, \\ 0 & \text{otherwise.} \end{cases}$$

Since $v^{\star\star} = v$, it follows from the general Hardy–Littlewood inequality (see [6]) that

$$\int_{\mathbb{R}^N} \frac{|u^{\star\star}|^q}{|y|^\beta} = \int_{\mathbb{R}^N} \left(|u|^q\right)^{\star\star} v^{\star\star} \geqslant \int_{\mathbb{R}^N} |u|^q v = \int_{\mathbb{R}^N} \frac{|u|^q}{|y|^\beta}.$$

The proof is complete. □

Remark 4. If $\beta < p$, then S is achieved (see [1]).

Remark 5. It is proved in [1], Theorem 5.3, that when $\beta < p = 2$, the optimal function for S satisfies $u = u^\star$.

Remark 6. The use of the "product" of two symmetrizations is applicable to many other problems. A simple example is given by constraints of the form

$$\int_{\mathbb{R}^N} \left|u(y, z)\right|^q g\left(|y|\right) h\left(|z|\right) = 1,$$

where g and h are non-increasing.

Added in proof

After completing this Note, we learned of the preprint [4], where the authors show that all minimizers for the Hardy–Sobolev inequality must have the aforementioned symmetry, after a possible translation in the z variable.

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