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# Clark formula and logarithmic Sobolev inequalities for Bernoulli measures ☆

# Formule de Clark et inégalités de Sobolev logarithmiques pour les mesures de Bernoulli

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**Abstract**

Using a Clark formula for the predictable representation of random variables in discrete time and adapting the method presented in [Electron. Commun. Probab. 2 (1997) 71–81] in the Brownian case, we obtain a proof of modified and $L^1$ logarithmic Sobolev inequalities for Bernoulli measures. We also prove a bound that improves these inequalities as well as the optimal constant inequality of [J. Funct. Anal. 156 (2) (1998) 347–365]. ***To cite this article: F. Gao, N. Privault, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***


**Résumé**

A l'aide d'une formule de Clark pour la représentation prévisible de variables aléatoire en temps discret et en adaptant la preuve présentée dans [Electron. Commun. Probab. 2 (1997) 71–81] dans le cas brownien, nous obtenons une preuve des inégalités de Sobolev logarithmiques (inégalité modifiée et inégalité $L^1$) pour les mesures de Bernoulli. Nous présentons aussi une borne qui améliore ces inégalités ainsi que l'inégalité de constante optimale de [J. Funct. Anal. 156 (2) (1998) 347–365]. ***Pour citer cet article : F. Gao, N. Privault, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***


## Version française abrégée

Le calcul stochastique et la formule de Clark ont été utilisés avec succès pour la preuve d'inégalités de Sobolev logarithmiques sur l'espace des chemins [3] et sur l'espace de Poisson [1,10]. Dans cette Note, des méthodes similaires sont utilisées en temps discret pour la preuve d'inégalités de Sobolev logarithmiques pour

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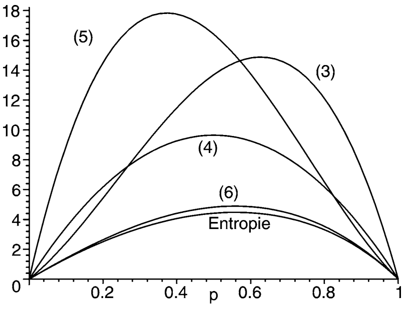


Fig. 1. Entropy as a function of $p$.

Fig. 1. L'entropie en fonction de $p$.

les mesures de Bernoulli. Par une autre méthode nous obtenons aussi une amélioration de ces inégalités. Soit $(X_k)_{k\in\mathbb{N}}$ une famille de variables aléatoires indépendantes de Bernoulli, à valeurs dans $\{-1,1\}$ sur $\Omega=\{-1,1\}^{\mathbb{N}}$, avec $p_k=P(X_k=1)>0$ et $q_k=P(X_k=-1)>0$, $k\in\mathbb{N}$. Dans [2], l'inégalité modifiée (3) a été prouvée pour $F$ de la forme $F=f(X_0,\ldots,X_n)$, avec le gradient modulo 2

$$\nabla_k F = X_k\big(f(X_0,\ldots,X_{k-1},-1,X_{k+1},\ldots,X_n)-f(X_0,\ldots,X_{k-1},1,X_{k+1},\ldots,X_n)\big),$$

$k\in\mathbb{N}$. Dans cette Note nous utilisons l'opérateur $D_k=-X_k\sqrt{p_kq_k}\nabla_k$, qui satisfait une formule de Clark, pour prouver (3) ainsi que (4) qui est une inégalité $L^1$. Nous prouvons aussi l'inégalité (6) :

$$\mathrm{Ent}\big[\mathrm{e}^F\big]\leqslant E\left[\mathrm{e}^F\sum_{k=0}^{k=N}p_kq_k\big(\nabla_kF\,\mathrm{e}^{\nabla_kF}-\mathrm{e}^{\nabla_kF}+1\big)\right]$$

qui améliore (3), (4) et (5), comme illustré sur le graphe, Fig. 1, en dimension un, où l'entropie est représentée comme fonction de $p\in[0,1]$, avec $f(1)=1$ et $f(-1)=3{,}5$.

L'inégalité (6) est en fait une version discrète de l'inégalité optimale prouvée dans [10] sur l'espace de Poisson.

## 1. Clark formula for Bernoulli measures

In this section we describe a chaotic calculus and its associated predictable representation formula in discrete time, cf. [9]. We also refer to [6,7], and [8] for other approaches to discrete chaotic calculus. Let $(X_k)_{k\in\mathbb{N}}$ be a family of i.i.d. Bernoulli random variables with $p_k=P(X_k=1)>0$, $q_k=P(X_k=-1)>0$, $k\in\mathbb{N}$, and let

$$Y_k=\sqrt{\frac{q_k}{p_k}}1_{\{X_k=1\}}-\sqrt{\frac{p_k}{q_k}}1_{\{X_k=-1\}},\quad k\in\mathbb{N},$$

so that $E[Y_k]=0$ and $E[Y_k^2]=1$, $k\in\mathbb{N}$. Let $\mathcal{F}_n=\sigma(X_0,\ldots,X_n)$, $n\in\mathbb{N}$, and $\mathcal{F}_{-1}=\{\emptyset,\Omega\}$. Let $(u_k)_{k\in\mathbb{N}}$ be a predictable process, i.e., $u_k$ is $\mathcal{F}_{k-1}$-measurable, $k\in\mathbb{N}$, and assume that $(u_k)_{k\in\mathbb{N}}$ is square-integrable. Let $\ell^2(\mathbb{N})^{\circ n}$ denote the space of square-summable symmetric functions in $n$ variables. Let

$$J_n(f_n)=\sum_{(k_1,\ldots,k_n)\in\Delta_n}f_n(k_1,\ldots,k_n)Y_{k_1}\cdots Y_{k_n},$$

$n\geqslant 1$, denote the discrete multiple stochastic integral of $f_n\in\ell^2(\mathbb{N})^{\circ n}$, where $\Delta_n=\{(k_1,\ldots,k_n)\in\mathbb{N}^n\colon k_i\neq k_j,\ 1\leqslant i<j\leqslant n\}$. Every $F\in L^2(\Omega,P)$ has an orthogonal decomposition

$$F=E[F]+\sum_{n=1}^{\infty}J_n(f_n),\quad f_n\in l^2(\mathbb{N})^{\circ n},\ n\geqslant 1.$$

We densely define the linear gradient operator $D : L^2(\Omega) \to L^2(\Omega \times \mathbb{N})$ as

$$D_k J_n(f_n) = n J_{n-1}\big(f_n(*, k) 1_{\Delta_n}(*, k)\big), \quad k \in \mathbb{N}, \ f_n \in \ell^2(\mathbb{N})^{\circ n}, \ n \in \mathbb{N}. \tag{1}$$

From (1) and the expression of $J_n(f_n)$ we obtain the probabilistic interpretation of $D_k$ as a finite difference operator:

$$D_k F = \sqrt{p_k q_k}\big(F_k^+ - F_k^-\big), \quad k \in \mathbb{N},$$

with $F = f(X_0, \dots, X_n)$ and

$$F_k^+ = f(X_0, \dots, X_{k-1}, +1, X_{k+1}, \dots, X_n), \qquad F_k^- = f(X_0, \dots, X_{k-1}, -1, X_{k+1}, \dots, X_n).$$

**Proposition 1.** *Let $F \in \mathrm{Dom}(D)$. We have the Clark formula*

$$F = E[F] + \sum_{k=0}^{\infty} E[D_k F \mid \mathcal{F}_{k-1}] Y_k. \tag{2}$$

**Proof.** The formula holds for $F = J_n(f_n)$:

$$\begin{aligned} J_n(f_n) &= n \sum_{k=0}^{\infty} J_{n-1}\big(f_n(*, k) 1_{[0,k-1]^{n-1}}(*)\big) Y_k = n \sum_{k=0}^{\infty} E\big[J_{n-1}\big(f_n(*, k)\big) \mid \mathcal{F}_{k-1}\big] Y_k \\ &= \sum_{k=0}^{\infty} E\big[D_k J_n(f_n) \mid \mathcal{F}_{k-1}\big] Y_k, \end{aligned}$$

and extends to $\mathrm{Dom}(D)$ by closability of $D$. □

## 2. Logarithmic Sobolev inequalities

We start by recovering the modified logarithmic Sobolev inequality of [2] from the Clark representation formula in discrete time.

**Theorem 1.** *Let $F \in \mathrm{Dom}(D)$ with $F > \eta$ for some $\eta > 0$. We have*

$$\mathrm{Ent}[F] \leqslant E\left[\frac{1}{F} \|DF\|^2_{\ell^2(\mathbb{N})}\right]. \tag{3}$$

**Proof.** Let $S_n = E[F \mid \mathcal{F}_n]$, $0 \leqslant n \leqslant N$. The Clark formula (2) reads

$$S_n = S_{-1} + \sum_{k=0}^{k=n} u_k Y_k,$$

with $u_k = E[D_k F \mid \mathcal{F}_{k-1}]$, $0 \leqslant k \leqslant n \leqslant N$, and $S_{-1} = E[F]$. Letting $f(x) = x \log x$ and using the bound

$$f(x+y) - f(x) = y \log x + (x+y) \log\left(1 + \frac{y}{x}\right) \leqslant y(1 + \log x) + \frac{y^2}{x},$$

we have:

$$\mathrm{Ent}[F] = E\big[f(S_N)\big] - E\big[f(S_{-1})\big] = E\left[\sum_{k=0}^{k=N} f(S_k) - f(S_{k-1})\right] = E\left[\sum_{k=0}^{k=N} f(S_{k-1} + Y_k u_k) - f(S_{k-1})\right]$$

$$
\begin{aligned}
&\leqslant E\left[\sum_{k=0}^{k=N} Y_k u_k(1+\log S_{k-1}) + \frac{Y_k^2 u_k^2}{S_{k-1}}\right] = E\left[\sum_{k=0}^{k=N} \frac{1}{E[F \mid \mathcal{F}_{k-1}]}\big(E[D_k F \mid \mathcal{F}_{k-1}]\big)^2\right] \\
&\leqslant E\left[\sum_{k=0}^{k=N} E\left[\frac{1}{F}(D_k F)^2 \mid \mathcal{F}_{k-1}\right]\right] = E\left[\frac{1}{F}\sum_{k=0}^{k=N}(D_k F)^2\right],
\end{aligned}
$$

where we used the convexity of $(u, v) \mapsto v^2/u$ as in the Wiener and Poisson cases [3] and [1]. The extension of this inequality to $F \in \mathrm{Dom}(D)$ relies on the closability of $D$. $\square$

The following $L^1$ inequality is present in [4] and [5], with applications to interacting random walks.

**Theorem 2.** *Let $F > 0$ be $\mathcal{F}_N$-measurable. We have*

$$
\mathrm{Ent}[F] \leqslant E\left[\sum_{k=0}^{k=N} D_k F D_k \log F\right]. \tag{4}
$$

**Proof.** Let $f(x) = x\log x$ and

$$
\Psi(x, y) = (x+y)\log(x+y) - x\log x - (1+\log x)y, \quad x, x+y > 0.
$$

From the relation

$$
\begin{aligned}
Y_k u_k &= Y_k E[D_k F \mid \mathcal{F}_{k-1}] = q_k 1_{\{X_k=1\}} E\big[(F_k^+ - F_k^-) \mid \mathcal{F}_{k-1}\big] + p_k 1_{\{X_k=-1\}} E\big[(F_k^- - F_k^+) \mid \mathcal{F}_{k-1}\big] \\
&= 1_{\{X_k=1\}} E\big[(F_k^+ - F_k^-)1_{\{X_k=-1\}} \mid \mathcal{F}_{k-1}\big] + 1_{\{X_k=-1\}} E\big[(F_k^- - F_k^+)1_{\{X_k=1\}} \mid \mathcal{F}_{k-1}\big],
\end{aligned}
$$

we have, using the convexity of $\Psi$:

$$
\begin{aligned}
\mathrm{Ent}[F] &= E\left[\sum_{k=0}^{k=N} f(S_{k-1} + Y_k u_k) - f(S_{k-1})\right] \\
&= E\left[\sum_{k=0}^{k=N} \Psi(S_{k-1}, Y_k u_k) + Y_k u_k(1+\log S_{k-1})\right] = E\left[\sum_{k=0}^{k=N} \Psi(S_{k-1}, Y_k u_k)\right] \\
&= E\left[\sum_{k=0}^{k=N} p_k \Psi\big(E[F \mid \mathcal{F}_{k-1}], E\big[(F_k^+ - F_k^-)1_{\{X_k=-1\}} \mid \mathcal{F}_{k-1}\big]\big)\right. \\
&\qquad \left. + q_k \Psi\big(E[F \mid \mathcal{F}_{k-1}], E\big[(F_k^- - F_k^+)1_{\{X_k=1\}} \mid \mathcal{F}_{k-1}\big]\big)\right] \\
&\leqslant E\left[\sum_{k=0}^{k=N} E\big[p_k \Psi\big(F, (F_k^+ - F_k^-)1_{\{X_k=-1\}}\big) + q_k \Psi\big(F, (F_k^- - F_k^+)1_{\{X_k=1\}}\big) \mid \mathcal{F}_{k-1}\big]\right] \\
&= E\left[\sum_{k=0}^{k=N} p_k 1_{\{X_k=-1\}} \Psi(F_k^-, F_k^+ - F_k^-) + q_k 1_{\{X_k=1\}} \Psi(F_k^+, F_k^- - F_k^+)\right] \\
&= E\left[\sum_{k=0}^{k=N} p_k q_k \Psi(F_k^-, F_k^+ - F_k^-) + p_k q_k \Psi(F_k^+, F_k^- - F_k^+)\right] \\
&= E\left[\sum_{k=0}^{k=N} D_k F D_k \log F\right]. \qquad \square
\end{aligned}
$$

Theorem 2 may also be proved by first using the bound

$$f(x+y)-f(x)=y\log x+(x+y)\log\left(1+\frac{y}{x}\right)\leqslant y(1+\log x)+y\log(x+y),$$

and then the convexity of $(u,v)\to v(\log(u+v)-\log u)$. The application of Theorem 2 to $\mathrm{e}^F$ for $\mathcal{F}_N$-measurable $F$ gives:

$$\begin{aligned}\mathrm{Ent}\left[\mathrm{e}^F\right] &\leqslant E\left[\sum_{k=0}^{k=N} p_k q_k \Psi\left(\mathrm{e}^{F_k^-},\mathrm{e}^{F_k^+}-\mathrm{e}^{F_k^-}\right)+p_k q_k \Psi\left(\mathrm{e}^{F_k^+},\mathrm{e}^{F_k^-}-\mathrm{e}^{F_k^+}\right)\right]\\ &= E\left[\sum_{k=0}^{k=N} p_k q_k\,\mathrm{e}^{F_k^-}\left(\nabla_K\,\mathrm{e}^{\nabla_k F}-\mathrm{e}^{\nabla_k F}+1\right)+p_k q_k\,\mathrm{e}^{F_k^+}\left(\nabla_k F\,\mathrm{e}^{\nabla F}-\mathrm{e}^{\nabla F}+1\right)\right],\end{aligned}$$

which is not comparable to the optimal constant inequality of [2]:

$$\mathrm{Ent}\left[\mathrm{e}^F\right]\leqslant E\left[\mathrm{e}^F\sum_{k=0}^{k=N} p_k q_k\left(|\nabla_k F|\,\mathrm{e}^{|\nabla_k F|}-\mathrm{e}^{|\nabla_k F|}+1\right)\right]. \tag{5}$$

The inequality (6) below is better than (4) and (5) (Fig. 1). It also improves (3) from the bound

$$x\,\mathrm{e}^x-\mathrm{e}^x+1\leqslant\left(\mathrm{e}^x-1\right)^2,\quad x\in\mathbb{R}.$$

**Theorem 3.** *Let $F$ be $\mathcal{F}_N$-measurable. We have*

$$\mathrm{Ent}\left[\mathrm{e}^F\right]\leqslant E\left[\mathrm{e}^F\sum_{k=0}^{k=N} p_k q_k\left(\nabla_k F\,\mathrm{e}^{\nabla_k F}-\mathrm{e}^{\nabla_k F}+1\right)\right]. \tag{6}$$

By tensorization the proof reduces to the following one-dimensional lemma.

**Lemma 1.** *For any $0\leqslant p\leqslant 1$, $t\in\mathbb{R}$, $a\in\mathbb{R}$, $q=1-p$,*

$$\begin{aligned}&pt\,\mathrm{e}^t+qa\,\mathrm{e}^a-\left(p\,\mathrm{e}^t+q\,\mathrm{e}^a\right)\log\left(p\,\mathrm{e}^t+q\,\mathrm{e}^a\right)\\ &\quad\leqslant pq\left(q\,\mathrm{e}^a\left((t-a)\,\mathrm{e}^{t-a}-\mathrm{e}^{t-a}+1\right)+p\,\mathrm{e}^t\left((a-t)\,\mathrm{e}^{a-t}-\mathrm{e}^{a-t}+1\right)\right).\end{aligned}$$

**Proof.** Set

$$\begin{aligned}g(t) &= pq\left(q\,\mathrm{e}^a\left((t-a)\,\mathrm{e}^{t-a}-\mathrm{e}^{t-a}+1\right)+p\,\mathrm{e}^t\left((a-t)\,\mathrm{e}^{a-t}-\mathrm{e}^{a-t}+1\right)\right)\\ &\quad -pt\,\mathrm{e}^t-qa\,\mathrm{e}^a+\left(p\,\mathrm{e}^t+q\,\mathrm{e}^a\right)\log\left(p\,\mathrm{e}^t+q\,\mathrm{e}^a\right).\end{aligned}$$

Then

$$g'(t)=pq\left(q\,\mathrm{e}^a(t-a)\,\mathrm{e}^{t-a}+p\,\mathrm{e}^t\left(-\mathrm{e}^{a-t}+1\right)\right)-pt\,\mathrm{e}^t+p\,\mathrm{e}^t\log\left(p\,\mathrm{e}^t+q\,\mathrm{e}^a\right)$$

and $g''(t)=p\,\mathrm{e}^t h(t)$, where

$$h(t)=-a-2pt-p+2pa+p^2t-p^2a+\log\left(p\,\mathrm{e}^t+q\,\mathrm{e}^a\right)+\frac{p\,\mathrm{e}^t}{p\,\mathrm{e}^t+q\,\mathrm{e}^a}.$$

Now,

$$h'(t)=\frac{pq^2(\mathrm{e}^t-\mathrm{e}^a)(p\,\mathrm{e}^t+(q+1)\,\mathrm{e}^a)}{(p\,\mathrm{e}^t+q\,\mathrm{e}^a)^2},$$

which implies that $h'(a) = 0$, $h'(t) < 0$ for any $t < a$ and $h'(t) > 0$ for any $t > a$. Hence, for any $t \neq a$, $h(t) > h(a) = 0$, and so $g''(t) \geqslant 0$ for any $t \in \mathbb{R}$ and $g''(t) = 0$ if and only if $t = a$. Therefore, $g'$ is strictly increasing. Finally, since $t = a$ is the unique root of $g' = 0$, we have that $g(t) \geqslant g(a) = 0$ for all $t \in \mathbb{R}$. □

In the symmetric case $p_k = q_k = 1/2$, $k \in \mathbb{N}$, (6) can be written:

$$\mathrm{Ent}\big[\mathrm{e}^F\big] \leqslant \frac{1}{2} E\left[\sum_{k=0}^{k=N} D_k F D \log F\right].$$

Let $U_n = (n + X_1 + \cdots + X_n)/2$, $F = \varphi(U_n)$, and $p_k = \lambda/n$, $k \in \mathbb{N}$, $\lambda > 0$. Then

$$\begin{aligned} &E\left[\mathrm{e}^F \sum_{k=0}^{k=n} p_k q_k \big(\nabla_k F\, \mathrm{e}^{\nabla_k F} - \mathrm{e}^{\nabla_k F} + 1\big)\right] \\ &\quad = \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right) U_n\, \mathrm{e}^{\varphi(U_n)} \big(\big(\varphi(U_n) - \varphi(U_n - 1)\big)\, \mathrm{e}^{\varphi(U_n) - \varphi(U_n - 1)} - \mathrm{e}^{\varphi(U_n) - \varphi(U_n - 1)} + 1\big) \\ &\qquad + \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)(n - U_n)\, \mathrm{e}^{\varphi(U_n)} \big(\big(\varphi(U_n + 1) - \varphi(U_n)\big)\, \mathrm{e}^{\varphi(U_n + 1) - \varphi(U_n)} - \mathrm{e}^{\varphi(U_n + 1) - \varphi(U_n)} + 1\big), \end{aligned}$$

and as $n$ goes to infinity we obtain from the Poisson limit theorem:

$$\mathrm{Ent}\big[\varphi(U)\big] \leqslant \lambda E\big[\mathrm{e}^{\varphi(U)} \big(\big(\varphi(U + 1) - \varphi(U)\big)\, \mathrm{e}^{\varphi(U + 1) - \varphi(U)} - \mathrm{e}^{\varphi(U + 1) - \varphi(U)} + 1\big)\big],$$

where $U$ is a Poisson random variable with parameter $\lambda$. This corresponds to the sharp inequality of [10]. However, (6) does not improve the condition $\sup_{k \in \mathbb{N}} |\nabla_k F| \leqslant \beta$ for the deviation result of [2], since the conditions $\nabla_k F \leqslant \beta$ a.e. and $|\nabla_k F| \leqslant \beta$ a.e. are equivalent, $k \in \mathbb{N}$.

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