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Differential Geometry

On bounds for total absolute curvature of surfaces in hyperbolic 3-space

Quelques bornes pour la courbure totale de surfaces dans l'espace hyperbolique de dimension 3

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Abstract

We construct examples of surfaces in hyperbolic space which do not satisfy the Chern–Lashof inequality (which holds for immersed surfaces in Euclidean space). ***To cite this article: R. Langevin, G. Solanes, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous construisons des exemples de surfaces dans l'espace hyperbolique qui ne satisfont pas l'inégalité de Chern–Lashof (qui est vérifiée pour les surfaces immergées dans l'espace euclidien). ***Pour citer cet article : R. Langevin, G. Solanes, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction

For an immersion of a closed surface M of genus g in Euclidean space, let K_e denote its Gauss curvature. Chern and Lashof proved in [1] that

$$\int_M |K_e| \geqslant 2\pi(2+2g). \tag{1}$$

To get an analog of this inequality for immersions in S^3, one must take into account not just the total absolute extrinsic curvature but the so called 1-length, and the area of M. More precisely (see [2]), for certain constants c_0, c_1, c_2, $\int_M c_2|K_e| + c_1 h_1 + c_0 \geqslant 2\pi(2+2g)$, where K_e denotes the extrinsic curvature and $h_1(p)$ is the average over all the directions of the absolute values of the normal curvatures in p.

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For immersed surfaces in hyperbolic space, it was stated in [3] that, if the surface M is contained in a ball of radius r, then $\int_M |K_e| > \frac{2\pi(2+2g)}{\cosh r}$. However, the inequality (1) was still expected to hold in the hyperbolic case [4].

For closed curves in hyperbolic space, a result better than the Fenchel–Fary–Milnor theorem for curves in $\mathbb{R}^3$ was proved in [5]. The inequality involves an extra area term. Let k be the geodesic curvature of a knot C in hyperbolic space. For every point $x \in C$, let A_x be the area of the surface defined by the segments joining x to all the points of C. For some point x, $\int |k|\,\mathrm{d}s \geqslant 4\pi + A_x$. In this paper, after giving some inequalities for total absolute extrinsic and intrinsic curvature of surfaces in hyperbolic space, we construct examples showing that the total absolute extrinsic curvature of a surface of genus g can not be bounded by just $2\pi(2+2g)$.

2. Extrinsic and intrinsic curvatures

Consider in $\mathbb{R}^4$ the Lorentz metric $L(x,y) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$. This defines a quadric Q in $\mathbb{RP}^3$. The Klein model of hyperbolic 3-space $\mathbb{H}^3$ is the interior of Q. Its metric corresponds to the restriction of L to the hyperboloid $\{x \in \mathbb{R}^4 \mid L(x,x) = -1\}$.

In this model, totally geodesic planes are intersections of projective planes with the interior of Q. By polarity with respect to Q, the space Λ of totally geodesic planes is identified to the exterior of Q. This space has a natural pseudo-Riemannian metric which corresponds to the restriction of L to the hyperboloid $\{x \in \mathbb{R}^4 \mid L(x,x) = 1\}$.

For an immersed smooth hypersurface $M \subset \mathbb{H}^3$, *total absolute curvature* is defined as $\int_M |K_e|$, where K_e is the extrinsic curvature of M. Note that, since $\mathbb{H}^3$ is of constant curvature -1, by the Gauss equation (see [6, p. 128]), the extrinsic curvature at a point p of M is $K_e = K_i + 1$, where K_i denotes the intrinsic curvature of M at p. Thus, if M is closed with genus g, the Gauss–Bonnet theorem gives

$$\int_M K_e = 2\pi(2-2g) + A(M), \tag{2}$$

where A denotes the area.

The surface M is smooth, but below we consider the boundary S of its convex hull which may not be smooth. Nevertheless, for *convex* hypersurfaces the total absolute curvature can be defined as follows. Let S be the boundary of a compact convex body $K \subset \mathbb{H}^3$ with nonempty interior and let $K^* \subset \Lambda$ be the set of planes that do not intersect the interior of K. The boundary S^* of K^* is the set of supporting planes of K. If O^* is the polar hyperplane of some point O interior to K, the affine chart $\mathbb{RP}^3 \setminus O^*$ contains both K and K^*. In suitable affine coordinates, Q is the unit sphere so that $K^* = \{\xi \mid \sup_K \langle \xi, \cdot\rangle \leqslant 1\}$ and it is convex. If S is smooth and strongly convex (with positive definite second fundamental form everywhere) then S^* is also smooth. In general, S^* is rectifiable and has a tangent plane almost everywhere. These planes are of spatial type in Λ since they do not meet Q. Therefore, we can consider the area measure A^* on S^* with respect to the metric of Λ. We define the total absolute curvature of S to be $A^*(S^*)$. From the results in [7], both definitions coincide in the case of smooth convex hypersurfaces.

Proposition 1. *Let S be the boundary of a compact convex body K. The total curvature of S is $A^*(S^*) = 4\pi + A(S)$.*

Proof. If S is smooth this is a particular case of (2). It is known that K can be approximated (in the Hausdorff metric) by a sequence (K_n) of strongly convex bodies with smooth boundary. Clearly the polar duals $K_n^* = \{\xi \mid \sup_{K_n} \langle \xi, \cdot\rangle \leqslant 1\}$ converge to K^*. We will see that $A^*(S_n^*)$ converge to $A^*(S^*)$ where $S_n^* = \partial K_n^*$.

Take a finite collection of compact domains T_i such that $\bigcup_i T_i = S^*$ and $T_i \cap T_j$ has null measure ($i \neq j$). For each i, fix u_i interior to T_i, consider the half-space containing u_i bounded by $V_i = \langle u_i\rangle^\perp$ and let π_i be the orthogonal projection of this half-space onto V_i. Taking the domains T_i small enough, there exist open convex sets $\Omega_i \subset V_i$ such that $S_n^* \cap \pi_i^{-1}(\Omega_i)$ is the graph of some smooth convex function f_n. Also, $S^* \cap \pi_i^{-1}(\Omega_i)$ is the graph of a convex function f. The functions f_n converge, uniformly on compact sets of Ω_i, to f (cf. [9, p. 90]). As a general fact on convex functions (cf. [8, p. 115] or [9, p. 248]), the differentials $\mathrm{d}f_n$ converge to $\mathrm{d}f$ almost everywhere and with uniform bound in each compact subset of Ω_i. Now, $U_i = \pi_i(T_i)$ is compact and, using dominated convergence, we have $\lim A^*(S_n^* \cap \pi_i^{-1}(U_i)) = A^*(S^* \cap \pi_i^{-1}(U_i))$ since the area of $S_n^* \cap \pi_i^{-1}(U_i)$ in $\Omega_i \times \mathbb{R}$ endowed with a pseudo-Riemannian metric can be expressed as, $A^*(S_n^* \cap \pi_i^{-1}(U_i)) = \int_{U_i} \sqrt{|P(x, f_n(x), \mathrm{d}f_n(x))|}$, where $P(x, y, z)$ is a polynomial in z of degree not greater than 2 whose coefficients are smooth functions on (x, y). To finish we must prove that $\lim A^*(S_n^*) = \lim \sum_i A^*(S_n^* \cap \pi_i^{-1}(U_i))$. However, this is clear since the measure of $\pi_i(\pi_i^{-1}(U_i) \cap \pi_j^{-1}(U_j) \cap S_n^*)$ goes to 0 and we have uniform bounds on compact sets. $\square$

The following proposition shows that (1) holds for topological spheres.

Proposition 2. *If M is a closed surface immersed in $\mathbb{H}^3$, then $\int_M |K_e| \geqslant 4\pi + A(S)$, where S is the boundary of the convex hull of M. The equality sign holds only when M is convex.*

Proof. Consider $U \subset M$ the relative interior of $M \cap S$ and $U^* \subset S^*$ consisting of the planes tangent to M at some point of U. From the results in [7], the total extrinsic curvature of U is the area of U^* with respect to the metric of Λ, $\int_U K_e = A^*(U^*)$. As it is known for Euclidean convex hulls, $S \setminus M$ is a generalized developable surface. This means that the supporting planes at points of $S \setminus M$ intersect S, at least, in some line segment. This fact implies (cf. [8, p. 115, (9.8)]) that the supporting planes at points of $S \setminus M$ form a null measure subset of S^*. Thus, U^* has full measure in S^*, $A^*(U^*) = A^*(S^*)$ and $\int_M K_e \geqslant \int_U K_e = A^*(S^*) = 4\pi + A(S)$. $\square$

For the intrinsic curvature, an analog to the Chern–Lashof theorem can be easily proved.

Proposition 3. *Let $M \subset \mathbb{H}^3$ be a closed surface of genus g immersed in the hyperbolic 3-space. Then $\int_M |K_i| \geqslant 2\pi(2 + 2g)$, where K_i is the intrinsic curvature of M. Equality holds only for topological spheres with nonnegative K_i.*

Proof. Let S be the boundary of the convex hull of M and U the relative interior of $M \cap S$. Set $K_i^+ = \max\{K_i, 0\}$ and $K_i^- = -\min\{K_i, 0\}$. From the proof of the last proposition,

$$\int_M K_i^+ \geqslant \int_U K_i^+ \geqslant \int_U K_i = \int_U (K_e - 1) = 4\pi + A(S) - A(U) \geqslant 4\pi. \tag{3}$$

On the other hand,

$$\int_M K_i = \int_M K_i^+ - \int_M K_i^- = 2\pi(2 - 2g). \tag{4}$$

Comparing (3) and (4) we get $\int_M K_i^- \geqslant 4\pi g$. $\square$

3. Examples of surfaces in $\mathbb{H}^3$

We will construct examples of surfaces showing that (1) does not hold in $\mathbb{H}^3$. Let us choose the affine chart $\mathbb{RP}^3 \setminus \{x^4 = 0\}$, now $\mathbb{H}^3$ is identified with the open unit ball $\mathbb{B}^3$ in $\mathbb{R}^3$. The corresponding metric can be written as

$$g = \frac{1}{(1 - r^2)^2}\,\mathrm{d}r^2 + \frac{r^2}{1 - r^2}\,\mathrm{d}\theta^2, \tag{5}$$

where (r, θ) are polar coordinates in the origin.

In this model, geodesic lines look like Euclidean chords of $\mathbb{B}^3$ and intersections of Euclidean planes with $\mathbb{B}^3$ are totally geodesic. As a consequence of this, in a point of a surface $M \subset \mathbb{B}^3$, the extrinsic curvature of M as an immersion in $\mathbb{H}^3$ and the curvature of M as a surface in $\mathbb{R}^3$ have the same sign. Indeed, the extrinsic curvature is negative if and only if the tangent totally geodesic plane intersects the surface locally in two transverse curves.

Consider the Euclidean cube $C = \{|x^i| \leqslant 1/2\} \subset \mathbb{B}^3$ (see Fig. 1). Modifying a small neighborhood of the corners of C we can get a convex domain C' with smooth boundary. For every n, drill in C', 4^n vertical Euclidean cylindrical holes with radius $1/(8 \cdot 2^n)$. The boundary of this domain is a non-smooth surface of genus 4^n. Modifying again a small neighborhood of the corners, we can get a smooth surface M_n such that all the points inside C' have nonpositive curvature for the Euclidean metric, therefore also for the hyperbolic metric.

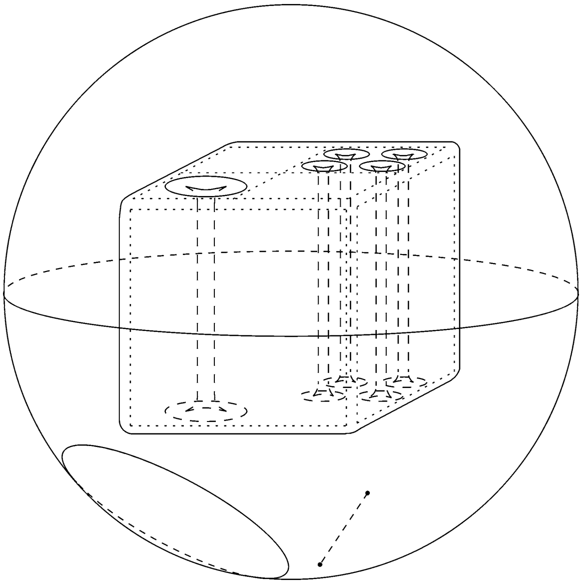

Fig. 1. Geodesic line, totally geodesic plane and the surface M_n in the Klein model.

Let K_e denote again the extrinsic curvature. If K_e^+ is the positive part of K_e,

$$\int_{M_n} |K_e| = 2\int_{M_n} K_e^+ - \int_{M_n} K_e.$$

Then, using (2)

$$\int_{M_n} |K_e| = 2\int_{\partial C'} K_e - 2\pi(2-2g) - A_n = 2(4\pi + A') - 2\pi(2-2g) - A_n = 2\pi(2+2g) + 2A' - A_n,$$

where A_n and A' denote, respectively, the areas of M_n and C'. Comparing (5) to the expression of Euclidean metric in polar coordinates, any vector of $T\mathbb{B}^3$ has a greater length with the hyperbolic metric than with the Euclidean one. Hence, the hyperbolic area of any surface will be also greater than the Euclidean area in this model. Since the Euclidean total area of the cylindrical holes is $2\pi(\frac{1}{2} - 2\varepsilon)\frac{1}{8\cdot 2^n} \cdot 4^n$, the Euclidean areas of M_n go to infinity and so do the hyperbolic areas. Then, for n big enough, $A_n > 2A'$, so the total absolute extrinsic curvature of M_n cannot be bounded by $2\pi(2+2g)$. It remains the question of deciding if (1) holds for tori.

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