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Support of Virasoro unitarizing measures

Hélène Airault [a], Paul Malliavin [b], Anton Thalmaier [c]

[a] INSSET, Université de Picardie, 48, rue Raspail, 02100 Saint-Quentin, France

[b] 10, rue Saint Louis en l'Isle, 75004 Paris, France

[c] IAM, Universität Bonn, Wegelerstr. 6, 53115 Bonn, Germany

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Abstract A unitarizing measure is a probability measure such that the associated L^2 space contains a closed subspace of holomorphic functionals on which the Virasoro algebra acts unitarily. It has been shown that the unitarizing property is equivalent to an a priori given formula of integration by parts, which has been computed explicitly. We show in this Note that unitarizing measures must be supported by the quotient of the homeomorphism group of the circle by the subgroup of Möbius transformations. ***To cite this article: H. Airault et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 621–626.***

Support des mesures unitarisantes de l'algèbre de Virasoro

Résumé Une mesure unitarisante de l'algèbre de Virasoro est une mesure de probabilité telle que l'espace L^2 associé contienne un sous-espace fermé de fonctionnelles holomorphes sur lequel l'algèbre de Virasoro agit de façon unitaire. On a caractérisé les mesures unitarisantes par une formule d'intégration par parties qui a été explicitement calculée. Dans cette Note on montre qu'une mesure unitarisante doit être portée par le quotient du groupe des homéomorphismes du cercle par le sous-groupe des transformations de Möbius. ***Pour citer cet article : H. Airault et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 621–626.***

Version française abrégée

Le mouvement brownien canonique sur le groupe des difféomorphismes du cercle est défini par la SDE

$$\mathrm{d}g_y(t) = \circ\,\mathrm{d}y(t)g_y(t), \quad y = \sum_{k>1} \varepsilon_k x_k, \quad \varepsilon_{2k} = \frac{1}{\sqrt{k^3-k}}\cos k\theta, \quad \varepsilon_{2k+1} = \frac{1}{\sqrt{k^3-k}}\sin k\theta,$$

où x_k dénote une suite infinie de mouvements browniens indépendants. Dans [6,5,3] il est montré que le brownien canonique peut être réalisé sur le groupe des homéomorphismes hölderiens du cercle. Une

E-mail addresses: hairault@insset.u-picardie.fr (H. Airault); sli@ccr.jussieu.fr (P. Malliavin); anton@wiener.iam.uni-bonn.de (A. Thalmaier).

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définition constructive d'une mesure unitarisante consiste à l'identifier avec les mesures invariantes pour la SDE

$$dg_y(t) = \big(\circ\, dy(t) - Z^\ell\, dt\big) g_y(t), \quad Z^\ell = \frac{1}{2}\sum_{k>1} \langle \varepsilon_k, \Omega\rangle \partial^\ell_{\varepsilon_k}, \tag{0.1}$$

où la forme différentielle Ω a été explicitement calculée dans [2, (2.2.5)]. Nous discuterons la réalisation des mesures unitarisantes dans le contexte étudié en [2] de deux réalisations des représentations de Virasoro : celle sur les fonctions univalentes et celle sur la grassmannienne de Sato. Dans ce dernier cas, est introduite sur la grassmannienne une forme kählerienne positive semi-définie ; pour obtenir une variété kählerienne, il est impératif de faire le quotient par le sous-espace isotrope ; la variété kählerienne ainsi obtenue constituera le support de la mesure unitarisante. Dans le premier cas, la forme différentielle définissant la formule d'intégration par parties s'annule sur une sous-algèbre de Lie isomorphe à $\mathrm{sl}(2;\mathbb{R})$. Comme la mesure de Haar de $\mathrm{SL}(2;\mathbb{R})$ est de masse totale infinie, il est impossible d'obtenir une mesure de probabilité satisfaisant la formule d'intégration par parties : il faut ainsi effectuer le quotient par $\mathrm{SL}(2;\mathbb{R})$; ce quotient portera alors la mesure unitarisante. Il est remarquable que, par des considérations différentes dans deux modèles distincts, on obtienne pour les mesures unitarisantes deux supports isomorphes.

1. Riemannian geometry on the Grassmannian model of Virasoro representations

1.1. The homogeneous Kähler manifold $\mathcal{M}_1$

We denote $\mathrm{Diff}(S^1)$ the group of C^∞ orientation preserving diffeomorphisms of the circle. The Kähler form used in [2] on $\mathcal{M} := \mathrm{Diff}(S^1)/S^1$ is not positive definite, see [4]. We shall remedy this situation by passing to a suitable quotient $\mathcal{M}_1$ for which $\mathcal{M}$ appears as a fiber space over $\mathcal{M}_1$. Denote by Γ the subgroup of $\mathrm{Diff}(S^1)$ constituted by the restriction to S^1 of Möbius transformations on the unit disk. A transformation in Γ is of the form

$$z \mapsto \alpha \frac{z-a}{1-\bar{a}z}, \quad z \in S^1,\ |\alpha| = 1,\ |a| < 1.$$

The Lie algebra $\mathrm{diff}(S^1)$ of left invariant vector fields on $\mathrm{Diff}(S^1)$ is identified with the vector fields $u(\theta)\frac{d}{d\theta}$ where u is a C^∞ function on S^1, the Lie bracket being given by $[u,v] = u\dot{v} - v\dot{u}$. Consider the three vectors $f_0 = 1,\ f_1 = \cos\theta,\ f_2 = \sin\theta$. We have the relations $[f_0, f_1] = -f_2$, $[f_0, f_2] = f_1$, $[f_1, f_2] = f_0$; therefore the vector space generated by $f_0,\ f_1,\ f_2$ constitutes a Lie subalgebra γ isomorphic to $\mathrm{sl}(2;\mathbb{R})$; we have $\Gamma = \exp(\gamma)$. We consider the cocyle on $\mathrm{diff}(S^1)$ defined by

$$\omega(u,v) = -\int_0^{2\pi} \big(u' + u^{(3)}\big) v \frac{d\theta}{2\pi}. \tag{1.1}$$

THEOREM 1.1. – (i) *The cocycle ω is invariant under the adjoint action of γ*; (ii) *The Hilbert transform J considered in* [2, (1.2.5)] *is invariant under the adjoint action of γ*; (iii) *denoting*

$$\mathcal{M}_1 := \mathrm{Diff}\big(S^1\big)/\,\Gamma,$$

then by passing to the quotient, J and ω define on $\mathcal{M}_1$ the structure of a homogeneous Kähler manifold. (iv) *The function K defined in* [2, 4.1] *satisfies*

$$K(gg_0) = K(g), \quad \forall g_0 \in \Gamma,$$

and therefore defines a function K_1 *on* $\mathcal{M}_1$;

$$6K_1 \text{ is the Kähler potential of } \mathcal{M}_1. \tag{1.2}$$

Proof. – Since $f_i''' + f_i' = 0$, we remark that $\omega(f_i, v) = 0$ for $i = 0, 1, 2$. Using the cocyle identity [2, (1.2.2)] we obtain that $\omega([f_i, u], v) + \omega(u, [f_i, v]) = 0$.

• We have to prove that $[f_i, J(u)] = J([f_i, u])$ for $u = \sum_{k\geqslant 2}(a_k \cos k\theta + b_k \sin k\theta)$. We check this relation case by case.

First take f_2 and $u = \cos k\theta$, then $2[f_2, Ju] = -2[f_2, \sin k\theta]$. By means of [2, p. 628], this equals $-(k-1)\sin(k+1)\theta + (k+1)\sin(k-1)\theta = J(-(k+1)\cos(k-1)\theta + (k-1)\cos(k+1)\theta) = 2J([f_2, \cos k\theta])$.

Secondly take f_2 and $u = \sin k\theta$, we get $2[f_2, Ju] = (k-1)\cos(k+1)\theta - (k+1)\cos(k-1)\theta = J((1-k)\sin(k+1)\theta + (k+1)\sin(k-1)\theta) = 2J([f_2, \sin k\theta])$.

• The action of f_1 is conjugate to the action of f_2 by a rotation of angle $\pi/2$: as rotations commute with the Hilbert transform, the result for f_2 implies the result for f_1; the action of f_0 is equivalent to taking derivative: $[f_0, u] = -u'$; as the Hilbert transform is a convolution operator, it commutes with the derivative.

• The operator U_g defined in [2, (3.1.2)] satisfies $U_{gg_0} = U_g \circ U_{g_0}$; furthermore the norm

$$\|u\|^2 := \sum_k k\big(a_k^2 + b_k^2\big) = \frac{1}{\pi}\int_D |\nabla h|^2 \,\mathrm{d}x \wedge \mathrm{d}y,$$

where h is the harmonic extension of u inside D. The invariance of the Dirichlet integral under conformal maps implies that U_{g_0} is a unitary transformation which preserves the subspace of antiholomorphic functions. Therefore, using the notations of [2, (4.1.4)], we have $b_{gg_0} b_{gg_0}^\dagger = b_g b_g^\dagger$, and we are left to use the identity $\det(\mathrm{I} + A) = \det(\mathrm{I} + A^*)$.

• Theorem 4.1.9 in [2] finally proves (1.2). □

1.2. Horizontal Laplacian above $\mathcal{M}_1$

We denote by $\mathcal{P} := \{u \in \mathrm{diff}(S^1) : \int u(\theta)\,\mathrm{d}\theta = \int u(\theta)\cos\theta\,\mathrm{d}\theta = \int u(\theta)\sin\theta\,\mathrm{d}\theta = 0\}$. The projection $\pi : \mathrm{Diff}(S^1) \to \mathcal{M}_1$ has a differential $\pi'(g)$ which restricted to $g\exp(\varepsilon\mathcal{P})$ realizes a surjective isomorphism; denote by ψ_g the inverse isomorphism. Given a tangent vector field Z on $\mathcal{M}_1$, its lift to $\mathrm{Diff}(S^1)$ is the vector field $\widetilde{Z}$ defined by $\widetilde{Z}_g = \psi_g(Z_{\pi(g)})$. We denote by $\omega = (\omega^k)$ the differential form on $\mathrm{Diff}(S^1)$ defined as the dual basis to the vector fields $\partial^r_{\varepsilon_*}$ where $\varepsilon_{2k} = \cos k\theta$, $\varepsilon_{2k-1} = \sin k\theta$, $k \geqslant 1$. We remark that $\omega_g = g^{-1}\,\mathrm{d}g$ is the left invariant Maurer Cartan differential form of $\mathrm{Diff}(S^1)$.

THEOREM 1.2. – *Given* Y, Z *two vector fields on* $\mathcal{M}_1$ *and* $m \in \mathcal{M}_1$, *denoting* ∇ *the Levi-Civita covariant derivative on* $\mathcal{M}_1$, *we have for* $g \in \pi^{-1}(m)$, $y := Y_m \in T_m(\mathcal{M}_1)$,

$$\psi_g(\nabla_y Z) - \sum_k \partial_{\psi_g(y)}\big(\big\langle \omega^k, \psi_*(Z)\big\rangle\big)\partial^r_{\varepsilon_k} = \sum_{i,k,s} \Gamma^i_{k,s}\big\langle \omega^k, \psi_g(Z_m)\big\rangle\big\langle \omega^s, \psi_g(y)\big\rangle \partial^r_{\varepsilon_i}, \tag{1.3}$$

where the Christoffel symbols $\Gamma^*_{*,*}$ *are given by*

$$\Gamma^i_{k,s} \equiv \frac{1}{2}(-\partial_i a_{k,s} + \partial_s a_{i,k} + \partial_k a_{s,i}) = \frac{1}{2}\big(\big([\varepsilon_i, \varepsilon_s] \mid \varepsilon_k\big) + \big([\varepsilon_i, \varepsilon_k] \mid \varepsilon_s\big)\big) \tag{1.4}$$

and $a_{*,*}$ *denote the components of the left invariant metric on* $\mathcal{M}_1$ *with respect to the exponential chart.*

Proof. – The covariant derivative is given by $\nabla_{\varepsilon_s}(\partial^r_{\varepsilon_k}) = \sum_i \Gamma^i_{s,k}\partial^r_{\varepsilon_i}$. We have to compute the Christoffel coefficients. As the metric is left invariant, it is sufficient to consider the situation at the identity element. We use the exponential chart $\Psi\colon \mathcal{P} \mapsto \mathcal{M}_1$ defined in a neighbourhhood of the identity by $z \mapsto \pi(\exp(z))$. The left-invariant Maurer–Cartan differential form has in the exponential chart the following expression (see [7], p. 24):

$$\omega_{\exp(z)} = \mathrm{I} - \tfrac{1}{2}\mathrm{ad}(z) + \mathrm{o}\big(|z|\big).$$

In the exponential chart Ψ and in terms of the injection $q : \mathcal{P} \to \mathfrak{g}$, the left invariant metric a on $\mathcal{M}_1$ writes as

$$a(z)(u,u) = \big\|\big(\mathrm{I} - \tfrac{1}{2}\mathrm{ad}(z)\big)q(u)\big\|^2_{\mathfrak{g}}, \quad u \in \mathcal{P},$$

which implies $(\partial_z a_{s,k})(0) = -\frac{1}{2}((\mathrm{ad}(z)\varepsilon_s \mid \varepsilon_k) + (\varepsilon_s \mid \mathrm{ad}(z)\varepsilon_k))$ and expression (1.4) for the Christoffel symbols. □

THEOREM 1.3. – *We have*

$$\sum_{k>1}\big((\nabla_k Z)^k - \big\langle \partial_{\varepsilon_k}\psi_g(Z), \omega^k\big\rangle\big) = 0. \tag{1.5}$$

Proof. – The statement is equivalent to $A_s := \sum_k \Gamma^k_{k,s} = \frac{1}{2}\sum_{k>1}([\varepsilon_k,\varepsilon_s] \mid \varepsilon_k) = 0$ for $s > 2$. Writing $[\varepsilon_k,\varepsilon_s] = \sum_\ell \beta(k,s,\ell)\varepsilon_\ell$, then $([\varepsilon_k,\varepsilon_s] \mid \varepsilon_k) = \beta(k,s,k)$, and the claim follows from the formulas in [2, p. 628] for the bracket in the trigonometric basis. □

We call *horizontal Laplacian* the differential operator on $\mathrm{Diff}(S^1)$ defined by

$$\Delta_H = \frac{1}{2}\sum_{k>1}\big(\partial^r_{\varepsilon_k}\big)^2,$$

see Fang [5] for the stochastic process associated to this horizontal Laplacian. To the Kähler metric on $\mathcal{M}_1$ the following Laplacian is associated: $2\Delta_{\mathcal{M}_1} f := -\mathrm{div}\,\mathrm{grad}(f) = \sum_k \nabla_{\varepsilon_k}\partial^r_{\varepsilon_k} f$. Expressing the covariant derivative by Christoffel symbols in the exponential chart we find that the drift Z at 0 satisfies $Z = \sum \varepsilon_s A_s$ and vanishes according to (1.5).

THEOREM 1.4. – *Given a functional* Φ *defined on* $\mathcal{M}_1$, *then* $\Delta_H(\Phi \circ \pi) = (\Delta_{\mathcal{M}_1}\Phi) \circ \pi$.

Proof. – We have $\Delta_H(\Phi \circ \pi) - (\Delta_{\mathcal{M}_1}\Phi) \circ \pi = \langle Z, d\Phi\rangle \circ \pi$. Take $g \in \pi^{-1}(m) \in G$ and consider the exponential chart $\exp_g : z \mapsto g\exp(\varepsilon z)$. We have $\Delta_{\mathcal{M}_1} = \sum_i \nabla^2_{e_i}$ and $\Delta_H = \sum_i \partial^2_i$. The claim then follows from $\nabla^2_{e_i} = \partial^2_i - \Gamma^k_{i,i}\partial_k$. □

COROLLARY 1.5. – *Consider on* $\mathrm{Diff}(S^1)$ *the process* $g_\omega(t)$ *associated to* Δ_H, *then* $t \mapsto \Phi(\pi(g_\omega(t)))$ *is a martingale for every holomorphic functional* Φ.

1.3. Unitarizing SDE on the Grassmannian

We denote by $\mathcal{J}$ the transformation of $\mathrm{Diff}(S^1)$ defined by $g \mapsto g^{-1}$, a map sending derivation on the left to derivation on the right.

THEOREM 1.6 (Transfer theorem). – *Let* $g_y(*)$ *be the process associated to the unitarizing SDE* (0.1), *then*

$$\gamma_y(t) := (\mathcal{J})\big(g_y(t)\big) \text{ is the process associated to } \frac{1}{2}\Delta_H - 3\sum_{k>1}\big(\partial^r_{\varepsilon_k}K\big)\partial^r_{\varepsilon_k}, \tag{1.6}$$

where K *is the defined in Theorem* 1.1; *furthermore*

$$m_y(t) := \pi\left(\gamma_y(t)\right) \text{ is the process associated to } \frac{1}{2}\Delta_{\mathcal{M}_1} - 3\,\nabla K_1 * \nabla, \tag{1.7}$$

where K_1 *is the Kähler potential of* $\mathcal{M}_1$ *defined in* (1.3); *finally the unitarizing drift defined in* (0.1) *can be written as*

$$Z^\ell = \frac{1}{2}\sum_{k>1}\left(\partial^r_{\varepsilon_k} K\right)\partial^r_{\varepsilon_k}. \tag{1.8}$$

Proof. – Apply [2], (4.2.5) and (4.1.5). □

Remark. – Introduce the horizontal Cauchy–Riemann operators on $\mathcal{M}_1$ defined as

$$\vartheta_\alpha := \partial^r_{2\alpha} - \sqrt{-1}\partial^r_{2\alpha+1}, \quad \vartheta_{\bar\alpha} := \bar\theta_\alpha;$$

then

$$\vartheta_{\bar\alpha}\vartheta_{\bar\beta}K \neq 0.$$

By means of the symmetry $\mathcal{I}$, we send this relation from the Grassmannian model to the model on univalent functions; using [2, (4.2.5)] we get $\vartheta_{\bar\beta}K = P_{\beta+1}$ where the sequence of Neretin polynomials P_* has been defined in [2, (2.1.1)]; then $\vartheta_{\bar\alpha}\vartheta_{\bar\beta}K = L^h_{-\beta}P_{\alpha+1}$ which cannot vanish identically according to identity [2, (A.7.1)]. The Hessian computed in terms of the complex Kählerian Christoffel symbols has the shape $\nabla^2_{\bar\alpha,\bar\beta}K = L^h_{-\beta}P_{\alpha+1} + \sum_{\lambda>0} c_\lambda P_\lambda$ where the constants c_λ are computed in terms of complex Christoffel symbols. Identities of the nature necessary for vanishing hold true for $\beta = 1$ according to [2, (A.7.3)]; on the other side it fails for $\beta = 2$, $\alpha = 1$; as a consequence the complex Hessian does not vanish.

2. Representation on univalent functions

The passage from left to right invariance realizes by Theorem 1.6 a transfer from the Grassmannian representation to the Kirillov representation acting from the right on the space $\mathcal{M}_u$ of univalent functions.

The group of Möbius transformations $\Gamma \simeq \mathrm{SL}(2;\mathbb{R})$ operates from the right on $\mathcal{M}_u$.

THEOREM 2.2. – *There does not exist a unitarizing measure on* $\mathcal{M}_u$.

Proof. – The complexified Lie algebra of γ is generated by L_{-1}, L_0, L_1. The unitarizing relations say that these three vector fields have divergence 0; desintegrating a unitarizing measure on $\mathcal{M}_u$ along equivalence classes modulo Γ, we obtain the Haar measure on $\mathrm{SL}(2;\mathbb{R})$ which has infinite volume. □

THEOREM 2.3. – *Consider the differential form* Ω *defined in* [2, (2.2.5)]:

$$\langle v, \Omega\rangle_f = \int_{S^1} z^2 S_f(z)v(z)\frac{\mathrm{d}z}{z} = \sum_{k>1}(P_k \circ \pi)\psi_k;$$

then Ω *is invariant under the Möbius action.*

Proof. – The L_0 invariance: we use [2], (2.2.4) and (2.1.1).

The L_1 invariance: we have $L^c_1 P_k = (k+1)P_{k-1}$; the Cartan formula gives $L^c_1\psi_k = \mathrm{i}(L_1)\,\mathrm{d}\psi_k$; formulas (2.2.3) and (2.2.1) in [2] yield $2L^c_1\psi_k = (-(k+2) - (2+k))\,\psi_{k+1} = -2(k+2)\,\psi_{k+1}$; finally

$$L^c_1(\Omega) = \sum_{k>1}(k+1)P_{k-1}\psi_k - \sum_{s>1}(s+2)P_s\psi_{s+1}. \tag{2.1}$$

Taking $k = s + 1$, all terms cancel except the term corresponding to $k = 2$ which vanishes because of $P_1 = 0$.

The L_{-1} invariance is proved in a similar way, this time using the identity $L^c_{-1} P_k = (k-1) P_{k+1}$ for $k > 1$, see [2, (A.7.3)]. □

We proceed as follows: by Cartan formula, $L^c_{-1}(\psi_k) = \mathrm{d}(i(L^c_{-1})\psi_k) + i(L^c_{-1})\,\mathrm{d}\psi_k$. For a form ψ of degree one, we have $i(X)\psi = \psi(X)$. Since $\psi_s(L^c_{-1}) = 0$ except if $s = 1$, we deduce that for $k > 1$, it holds $i(L^c_{-1})\psi_k = \psi_k(L^c_{-1}) = 0$. Thus, $2L^c_{-1}(\psi_k) = 2i(L^c_{-1})\,\mathrm{d}\psi_k$. With formula (2.2.3) in [2], together with $i(X)(\omega \wedge \omega') = (i(X)\omega) \wedge \omega' + (-1)^r \omega \wedge (i(X)\omega')$ if ω is a form of degree r, we deduce that

$$2i\left(L^c_{-1}\right)\mathrm{d}\psi_k = -\sum_{s \in Z}(k+2s)\left(i\left(L^c_{-1}\right)\psi_{-s}\right)\psi_{k+s} + \sum_{s \in Z}(k+2s)\left(i\left(L^c_{-1}\right)\psi_{k+s}\right)\psi_{-s}.$$

Applying again $i(L^c_{-1})\psi_s = 0$ except if $s = 1$ and $i(L^c_{-1})\psi_1 = 1$, we find that $2i(L^c_{-1})\,\mathrm{d}\psi_k = -(k-2) \times \psi_{k-1} + (2-k)\psi_{k-1}$. This yields $L^c_{-1}(\psi_k) = (2-k)\psi_{k-1}$. With the expression of Ω given in (2.2.4), [2], we calculate

$$\begin{aligned} L^c_{-1}\Omega &= \sum_{k>1}\left[L^c_{-1}(P_k)\psi_k + P_k(2-k)\psi_{k-1}\right] \\ &= \sum_{k>1}\left[(k-1)P_{k+1}\psi_k + P_k(2-k)\psi_{k-1}\right] \\ &= \sum_{k>1}(k-1)P_{k+1}\psi_k + \sum_{k>2} P_k(2-k)\psi_{k-1}. \end{aligned} \tag{2.2}$$

We put $k = h + 1$ in the last expression. This gives $L^c_{-1}\Omega = 0$.

Remark. – In [1] a probability measure μ has been constructed which has divergence zero relatively to the vector fields L^h_1, L^h_{-1}. This measure is constructed as a limit of probability measures μ_n supported by $\mathbb{C}^n$. Each measure μ_n satisfies an approximative formula of integration by parts which is established by using an integration by parts on the whole $\mathbb{C}^n$. According to De Branges theorem the domain of integration on which μ_n has to be considered is an open subset B_n on $\mathbb{C}^n$ which is defined as the range of the first Taylor coefficients of normalized univalent functions. Therefore the procedure used in [1] will give rise to a non-vanishing boundary term carried by ∂B_n: hence the method of [1] is not localizable on the space $\mathcal{M}$.

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