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A surface is a continuous function of its two fundamental forms

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Abstract If a field of positive definite symmetric matrices of order two and a field of symmetric matrices of order two together satisfy the Gauß and Codazzi–Mainardi equations in a connected and simply connected open subset of $\mathbb{R}^2$, then these fields are the first and second fundamental forms of a surface in $\mathbb{R}^3$, unique up to isometries. It is shown here that a surface defined in this fashion varies continuously as a function of its two fundamental forms, for *ad hoc* metrizable topologies. ***To cite this article: P.G. Ciarlet, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 609–614.***

Une surface est une fonction continue de ses deux formes fondamentales

Résumé Si un champ de matrices symétriques définies positives d'ordre deux et un champ de matrices symétriques d'ordre deux vérifient ensemble les équations de Gauß et de Codazzi–Mainardi dans un ouvert connexe et simplement connexe de $\mathbb{R}^2$, alors ces champs sont les première et deuxième formes fondamentales d'une surface dans $\mathbb{R}^3$, unique aux isométries près. On établit ici qu'une surface définie de cette façon varie continûment en fonction de ses deux formes fondamentales, pour des topologies métrisables convenables. ***Pour citer cet article : P.G. Ciarlet, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 609–614.***

1. Formulation of the problem

All spaces, matrices, etc., considered are real. The notations $\mathbb{M}^d$, $\mathbb{O}^d$, $\mathbb{S}^d$, and $\mathbb{S}^d_>$ respectively designate the sets of all square matrices of order d, of all orthogonal matrices of order d, of all symmetric matrices of order d, and of all symmetric and positive definite matrices of order d.

Latin indices and exponents vary in the set $\{1, 2, 3\}$ except when they are used for indexing sequences or when otherwise indicated, Greek indices and exponents vary in the set $\{1, 2\}$, and the summation convention with respect to repeated indices or exponents is used in conjunction with these rules. Kronecker's symbols are designated by δ_{ij} or δ_i^j according to the context.

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Let $\mathbf{E}^3$ denote a three-dimensional Euclidean space, let $\boldsymbol{a}\cdot\boldsymbol{b}$ denote the Euclidean inner product of $\boldsymbol{a},\boldsymbol{b}\in\mathbf{E}^3$, and let $|\boldsymbol{a}|=\sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$ denote the Euclidean norm of $\boldsymbol{a}\in\mathbf{E}^3$. Let $\rho(\mathbf{A})$ denote the spectral radius and let $|\mathbf{A}|:=\{\rho(\mathbf{A}^T\mathbf{A})\}^{1/2}$ denote the spectral norm of a matrix $\mathbf{A}\in\mathbb{M}^3$.

Let there be given a *two-dimensional vector space*, identified with $\mathbb{R}^2$. Let y_α denote the coordinates of a point $y\in\mathbb{R}^2$ and let $\partial_\alpha:=\partial/\partial y_\alpha$ and $\partial_{\alpha\beta}:=\partial^2/\partial y_\alpha\partial y_\beta$. Let ω be an open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta}\in\mathcal{C}^2(\omega;\mathbf{E}^3)$ be an *immersion*. The image $\boldsymbol{\theta}(\omega)$ is a *surface* in $\mathbf{E}^3$. The *first* and *second fundamental forms of the surface* $\boldsymbol{\theta}(\omega)$ are defined by means of their *covariant components*

$$a_{\alpha\beta}(y):=\partial_\alpha\boldsymbol{\theta}(y)\cdot\partial_\beta\boldsymbol{\theta}(y)\quad\text{and}\quad b_{\alpha\beta}(y):=\partial_{\alpha\beta}\boldsymbol{\theta}(y)\cdot\left\{\frac{\partial_1\boldsymbol{\theta}(y)\wedge\partial_2\boldsymbol{\theta}(y)}{|\partial_1\boldsymbol{\theta}(y)\wedge\partial_2\boldsymbol{\theta}(y)|}\right\},\quad y\in\omega.$$

We now recall two classical results from differential geometry, which are essential to the ensuing analysis. Theorem 1.1 provides sufficient conditions guaranteeing that, given two smooth enough matrix fields $(a_{\alpha\beta}):\omega\to\mathbb{S}^2_>$ and $(b_{\alpha\beta}):\omega\to\mathbb{S}^2$, there exists an immersion $\boldsymbol{\theta}:\omega\to\mathbf{E}^3$ such that these fields are the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$. Theorem 1.2 specifies how two such immersions differ (a self-contained, complete, and essentially elementary, proof of these well-known results, which together constitute the *fundamental theorem of surface theory*, is found in Ciarlet and Larsonneur [2]; a direct proof of the fundamental theorem of surface theory is given in Klingenberg [5, Theorem 3.8.8]).

THEOREM 1.1 (Global Existence Theorem). – *Let ω be a connected and simply connected open subset of $\mathbb{R}^2$ and let $(a_{\alpha\beta})\in\mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta})\in\mathcal{C}^2(\omega;\mathbb{S}^2)$ be two matrix fields that satisfy the Gauß and Codazzi–Mainardi equations, viz.,*

$$\partial_\beta C_{\alpha\sigma\tau}-\partial_\sigma C_{\alpha\beta\tau}+C^\mu_{\alpha\beta}C_{\sigma\tau\mu}-C^\mu_{\alpha\sigma}C_{\beta\tau\mu}=b_{\alpha\sigma}b_{\beta\tau}-b_{\alpha\beta}b_{\sigma\tau}\quad\text{in }\omega,$$

$$\partial_\beta b_{\alpha\sigma}-\partial_\sigma b_{\alpha\beta}+C^\mu_{\alpha\sigma}b_{\beta\mu}-C^\mu_{\alpha\beta}b_{\sigma\mu}=0\quad\text{in }\omega,$$

where

$$C_{\alpha\beta\tau}:=\frac{1}{2}(\partial_\beta a_{\alpha\tau}+\partial_\alpha a_{\beta\tau}-\partial_\tau a_{\alpha\beta})\quad\text{and}\quad C^\sigma_{\alpha\beta}:=a^{\sigma\tau}C_{\alpha\beta\tau},\quad\text{where }\left(a^{\sigma\tau}\right):=(a_{\alpha\beta})^{-1}.$$

Then there exists an immersion $\boldsymbol{\theta}\in\mathcal{C}^3(\omega;\mathbf{E}^3)$ such that

$$a_{\alpha\beta}=\partial_\alpha\boldsymbol{\theta}\cdot\partial_\beta\boldsymbol{\theta}\quad\text{and}\quad b_{\alpha\beta}=\partial_{\alpha\beta}\boldsymbol{\theta}\cdot\left\{\frac{\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}|}\right\}\quad\text{in }\omega.\qquad\square$$

THEOREM 1.2 (Rigidity Theorem). – *Let ω be a connected open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta}\in\mathcal{C}^2(\omega;\mathbf{E}^3)$ and $\widetilde{\boldsymbol{\theta}}\in\mathcal{C}^2(\omega;\mathbf{E}^3)$ be two immersions such that their associated first and second fundamental forms satisfy* (*with self-explanatory notations*) *$a_{\alpha\beta}=\widetilde{a}_{\alpha\beta}$ and $b_{\alpha\beta}=\widetilde{b}_{\alpha\beta}$ in ω. Then there exist a vector $\boldsymbol{a}\in\mathbf{E}^3$ and an orthogonal matrix $\mathbf{Q}\in\mathbb{O}^3$ such that $\boldsymbol{\theta}(y)=\boldsymbol{a}+\mathbf{Q}\widetilde{\boldsymbol{\theta}}(y)$ for all $y\in\omega$.* $\square$

Together, Theorems 1.1 and 1.2 establish the existence of a mapping F that associates to any matrix fields $(a_{\alpha\beta})\in\mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta})\in\mathcal{C}^2(\omega;\mathbb{S}^2)$ satisfying the Gauß and Codazzi–Mainardi equations in ω a well-defined element $F((a_{\alpha\beta}),(b_{\alpha\beta}))$ in the quotient set $\mathcal{C}^3(\omega;\mathbf{E}^3)/R$, where $(\boldsymbol{\theta},\widetilde{\boldsymbol{\theta}})\in R$ means that there exists a vector $\boldsymbol{a}\in\mathbf{E}^3$ and a matrix $\mathbf{Q}\in\mathbb{O}^3$ such that $\boldsymbol{\theta}(y)=\boldsymbol{a}+\mathbf{Q}\widetilde{\boldsymbol{\theta}}(y)$ for all $y\in\omega$.

A natural question thus arises as to whether there exist *ad hoc* topologies on the space $\mathcal{C}^2(\omega;\mathbb{S}^2_>)\times\mathcal{C}^2(\omega;\mathbb{S}^2)$ and on the quotient set $\mathcal{C}^3(\omega;\mathbf{E}^3)/R$ such that the mapping F defined in this fashion is *continuous*.

2. The analogous problem in "dimension three"

The purpose of this Note is to provide an affirmative answer to the above question through a proof that relies in an essential way on the solution to an *analogous question "in dimension three"*, as given in Ciarlet and Laurent [3,4]. In this section, we accordingly briefly review this analog problem.

Let there be given a three-dimensional vector space, identified with $\mathbb{R}^3$. Let x_i denote the coordinates of a point $x \in \mathbb{R}^3$ and let $\partial_i := \partial/\partial x_i$. Let Ω be an open subset of $\mathbb{R}^3$. The notation $K \Subset \Omega$ means that K is a compact subset of Ω. If $\mathbf{\Theta} \in \mathcal{C}^\ell(\Omega; \mathbf{E}^3)$ or $\mathbf{A} \in \mathcal{C}^\ell(\Omega; \mathbb{M}^3)$, $\ell \geqslant 0$, and $K \Subset \Omega$, we let $\|\mathbf{\Theta}\|_{\ell,K} := \sup_{\{x\in K, |\alpha|\leqslant \ell\}} |\partial^\alpha \mathbf{\Theta}(x)|$ and $\|\mathbf{A}\|_{\ell,K} := \sup_{\{x\in K, |\alpha|\leqslant \ell\}} |\partial^\alpha \mathbf{A}(x)|$, where ∂^α stands for the standard multi-index notation for partial derivatives, and $|\cdot|$ denotes the Euclidean vector norm or the matrix spectral norm.

Let $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ be an *immersion*. Then the *metric tensor field* $(g_{ij}) \in \mathcal{C}^0(\Omega; \mathbb{S}^3_>)$ of the set $\mathbf{\Theta}(\Omega)$ is defined by means of its *covariant components* $g_{ij}(x) := \partial_i \mathbf{\Theta}(x) \cdot \partial_j \mathbf{\Theta}(x)$, $x \in \Omega$. In geometrically exact three-dimensional elasticity, the matrix $(g_{ij}(x))$ is usually denoted $\mathbf{C}(x) := (g_{ij}(x))$, and is called the (right) *Cauchy–Green tensor at x*.

We now recall two classical results from three–dimensional differential geometry. Theorem 2.1 provides sufficient conditions guaranteeing that, given a smooth enough matrix field $\mathbf{C} = (g_{ij}) : \Omega \to \mathbb{S}^3_>$, there exists an immersion $\mathbf{\Theta} : \Omega \to \mathbf{E}^3$ such that $\mathbf{C}$ is the metric tensor field of the set $\mathbf{\Theta}(\Omega)$, while Theorem 2.2 specifies how two such immersions differ.

THEOREM 2.1 (Global Existence Theorem). – *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$ and let $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ be a matrix field that satisfies*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij} \Gamma_{kqp} - \Gamma^p_{ik} \Gamma_{jqp} = 0 \quad \text{in } \Omega,$$

where

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \quad \text{and} \quad \Gamma^p_{ij} := g^{pq} \Gamma_{ijq}, \quad \text{where } \left(g^{pq}\right) := (g_{ij})^{-1}.$$

Then there exists an immersion $\mathbf{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ such that $\mathbf{C} = \nabla\mathbf{\Theta}^{\mathrm{T}} \nabla\mathbf{\Theta}$ in Ω. □

THEOREM 2.2 (Rigidity Theorem). – *Let Ω be a connected open subset of $\mathbb{R}^3$ and let $\mathbf{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ and $\widetilde{\mathbf{\Theta}} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ be two immersions whose associated metric tensors $\mathbf{C} = \nabla\mathbf{\Theta}^{\mathrm{T}} \nabla\mathbf{\Theta}$ and $\widetilde{\mathbf{C}} = \nabla\widetilde{\mathbf{\Theta}}^{\mathrm{T}} \nabla\widetilde{\mathbf{\Theta}}$ satisfy $\mathbf{C} = \widetilde{\mathbf{C}}$ in Ω. Then there exist a vector $\boldsymbol{a} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3$ such that $\mathbf{\Theta}(x) = \boldsymbol{a} + \mathbf{Q}\widetilde{\mathbf{\Theta}}(x)$ for all $x \in \Omega$.* □

Together, Theorems 2.1 and 2.2 establish the existence of a mapping $\mathcal{F}$ that associates to any matrix field $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ satisfying $R_{qijk} = 0$ in Ω a well-defined element $\mathcal{F}(\mathbf{C})$ in the quotient set $\mathcal{C}^3(\Omega; \mathbf{E}^3)/\mathcal{R}$, where $(\mathbf{\Theta}, \widetilde{\mathbf{\Theta}}) \in \mathcal{R}$ means that there exists a vector $\boldsymbol{a} \in \mathbf{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3$ such that $\mathbf{\Theta}(x) = \boldsymbol{a} + \mathbf{Q}\widetilde{\mathbf{\Theta}}(x)$ for all $x \in \Omega$.

As shown by Ciarlet and Laurent [3,4], the continuity of the mapping $\mathcal{F}$ for *ad hoc* topologies on the space $\mathcal{C}^2(\Omega; \mathbb{S}^3)$ and on the quotient set $\mathcal{C}^3(\Omega; \mathbf{E}^3)/\mathcal{R}$ is a consequence of the following crucial result, which likewise plays a key role here (see the proof of Lemma 3.6 below).

THEOREM 2.3. – *Let Ω be a connected and simply connected open subset of $\mathbb{R}^3$. Let $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, and $\mathbf{C}^n = (g^n_{ij}) \in \mathcal{C}^2(\Omega, \mathbb{S}^3_>)$, $n \geqslant 0$, be matrix fields respectively satisfying $R_{qijk} = 0$ in Ω and $R^n_{qijk} = 0$ in Ω, $n \geqslant 0$ (with self-explanatory notations), such that $\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0$ for all $K \Subset \Omega$. Let $\mathbf{\Theta} \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ be any mapping that satisfies $\nabla\mathbf{\Theta}^{\mathrm{T}} \nabla\mathbf{\Theta} = \mathbf{C}$ in Ω (such mappings exist by Theorem 2.1). Then there exist mappings $\mathbf{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ satisfying $(\nabla\mathbf{\Theta}^n)^{\mathrm{T}} \nabla\mathbf{\Theta}^n = \mathbf{C}^n$ in Ω, $n \geqslant 0$, such that $\lim_{n\to\infty} \|\mathbf{\Theta}^n - \mathbf{\Theta}\|_{3,K} = 0$ for all $K \Subset \Omega$.* □

3. Main results

Let ω be an open subset of $\mathbb{R}^3$. The notation $\kappa \Subset \omega$ means that κ is a compact subset of ω. If $f \in \mathcal{C}^\ell(\omega;\mathbb{R})$ or $\boldsymbol{\theta} \in \mathcal{C}^\ell(\omega;\mathbf{E}^3)$, $\ell \geqslant 0$, and $\kappa \Subset \omega$, we let $\|f\|_{\ell,\kappa} := \sup_{\left\{\substack{y\in\kappa\\ |\alpha|\leqslant\ell}\right.} |\partial^\alpha f(y)|$ and $\|\boldsymbol{\theta}\|_{\ell,\kappa} := \sup_{\left\{\substack{y\in\kappa\\ |\alpha|\leqslant\ell}\right.} |\partial^\alpha \boldsymbol{\theta}(y)|$. If $\mathbf{A} \in \mathcal{C}^\ell(\omega;\mathbb{M}^d)$, $\ell \geqslant 0$, $d=2$ or 3, and $\kappa \Subset \omega$, we let $\|\mathbf{A}\|_{\ell,\kappa} = \sup_{\left\{\substack{y\in\kappa\\ |\alpha|\leqslant\ell}\right.} |\partial^\alpha \mathbf{A}(y)|$.

The next theorem constitutes the key step towards establishing the continuity of a surface as a function of its two fundamental forms (see Theorem 3.7).

THEOREM 3.1. – *Let ω be a connected and simply connected open subset of $\mathbb{R}^2$. Let $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2)$ be matrix fields satisfying the Gauß and Codazzi–Mainardi equations in ω and let $(a^n_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and $(b^n_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2)$ be matrix fields satisfying for each $n \geqslant 0$ the Gauß and Codazzi–Mainardi equations in ω. Assume that these matrix fields satisfy*

$$\lim_{n\to\infty}\left\|a^n_{\alpha\beta}-a_{\alpha\beta}\right\|_{2,\kappa}=0 \quad \textit{and} \quad \lim_{n\to\infty}\left\|b^n_{\alpha\beta}-b_{\alpha\beta}\right\|_{2,\kappa}=0 \quad \textit{for all } \kappa\Subset\omega.$$

Let $\boldsymbol{\theta} \in \mathcal{C}^3(\omega;\mathbf{E}^3)$ be any mapping that satisfies

$$a_{\alpha\beta}=\partial_\alpha\boldsymbol{\theta}\cdot\partial_\beta\boldsymbol{\theta} \quad \textit{and} \quad b_{\alpha\beta}=\partial_{\alpha\beta}\boldsymbol{\theta}\cdot\left\{\frac{\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}|}\right\} \quad \textit{in } \omega$$

(*such mappings exist by Theorem* 1.1). *Then there exist mappings $\boldsymbol{\theta}^n \in \mathcal{C}^3(\omega;\mathbf{E}^3)$ satisfying*

$$a^n_{\alpha\beta}=\partial_\alpha\boldsymbol{\theta}^n\cdot\partial_\beta\boldsymbol{\theta}^n \quad \textit{and} \quad b^n_{\alpha\beta}=\partial_{\alpha\beta}\boldsymbol{\theta}^n\cdot\left\{\frac{\partial_1\boldsymbol{\theta}^n\wedge\partial_2\boldsymbol{\theta}^n}{|\partial_1\boldsymbol{\theta}^n\wedge\partial_2\boldsymbol{\theta}^n|}\right\} \quad \textit{in } \omega, \quad n\geqslant 0,$$

such that $\lim_{n\to\infty}\|\boldsymbol{\theta}^n-\boldsymbol{\theta}\|_{3,\kappa}=0$ *for all* $\kappa\Subset\omega$. □

For clarity, the proof of Theorem 3.1 is broken into a series of five lemmas, the proofs of which are only briefly sketched here. Complete proofs are found in Ciarlet [1].

LEMMA 3.2. – *Let the matrix fields $(g_{ij}) \in \mathcal{C}^2(\omega\times\mathbb{R};\mathbb{S}^3)$ and $(g^n_{ij}) \in \mathcal{C}^2(\omega\times\mathbb{R};\mathbb{S}^3)$, $n\geqslant 0$, be defined by*

$$g_{\alpha\beta} := a_{\alpha\beta}-2x_3b_{\alpha\beta}+x_3^2c_{\alpha\beta} \quad \textit{and} \quad g_{i3}:=\delta_{i3},$$

$$g^n_{\alpha\beta} := a^n_{\alpha\beta}-2x_3b^n_{\alpha\beta}+x_3^2c^n_{\alpha\beta} \quad \textit{and} \quad g^n_{i3}:=\delta_{i3}, \quad n\geqslant 0$$

(*the variable $y\in\omega$ is omitted, x_3 designates the variable in* $\mathbb{R}$), *where*

$$c_{\alpha\beta}:=b^\tau_\alpha b_{\beta\tau}, \quad b^\tau_\alpha:=a^{\sigma\tau}b_{\alpha\sigma}, \quad \left(a^{\sigma\tau}\right):=(a_{\alpha\beta})^{-1},$$

$$c^n_{\alpha\beta}:=b^{\tau,n}_\alpha b^n_{\beta\tau}, \quad b^{\tau,n}_\alpha:=a^{\sigma\tau,n}b^n_{\alpha\sigma}, \quad \left(a^{\sigma\tau,n}\right):=\left(a^n_{\alpha\beta}\right)^{-1}, \quad n\geqslant 0.$$

Let ω_0 be an open subset of $\mathbb{R}^2$ such that $\overline{\omega}_0\Subset\omega$. Then there exists $\varepsilon_0=\varepsilon_0(\omega_0)>0$ such that the symmetric matrices $\mathbf{C}(y,x_3):=(g_{ij}(y,x_3))$ and $\mathbf{C}^n(y,x_3):=(g^n_{ij}(y,x_3))$, $n\geqslant 0$, are positive definite at all points $(y,x_3)\in\overline{\Omega}_0$, where $\Omega_0:=\omega_0\times\,]-\varepsilon_0,\varepsilon_0[$.

Sketch of proof. – The matrices $\mathbf{C}(y,x_3):=(g_{ij}(y,x_3))\in\mathbb{S}^3$ and $\mathbf{C}^n(y,x_3):=(g^n_{ij}(y,x_3))\in\mathbb{S}^3$ are of the form:

$$\mathbf{C}(y,x_3)=\mathbf{C}_0(y)+x_3\mathbf{C}_1(y)+x_3^2\mathbf{C}_2(y) \quad \text{and} \quad \mathbf{C}^n(y,x_3)=\mathbf{C}^n_0(y)+x_3\mathbf{C}^n_1(y)+x_3^2\mathbf{C}^n_2(y), \quad n\geqslant 0.$$

First, it is deduced from the assumptions $\lim_{n\to\infty}\|a^n_{\alpha\beta}-a_{\alpha\beta}\|_{0,\overline{\omega}_0}=0$ and $\lim_{n\to\infty}\|b^n_{\alpha\beta}-b_{\alpha\beta}\|_{0,\overline{\omega}_0}=0$ that there exists a constant M such that

$$\left\|\left(\mathbf{C}^n_0\right)^{-1}\right\|_{0,\overline{\omega}_0}+\left\|\mathbf{C}^n_1\right\|_{0,\overline{\omega}_0}+\left\|\mathbf{C}^n_2\right\|_{0,\overline{\omega}_0}\leqslant M \quad \text{for all } n\geqslant 0.$$

This uniform bound implies in turn that there exists $\varepsilon_0 = \varepsilon_0(\omega_0) > 0$ such that the matrices $\mathbf{C}(y, x_3)$ and $\mathbf{C}^n(y, x_3)$, $n \geqslant 0$, are invertible for all $(y, x_3) \in \overline{\omega}_0 \times [-\varepsilon_0, \varepsilon_0]$. Since these matrices are positive definite for $x_3 = 0$ by assumption, they remain so for all $x_3 \in [-\varepsilon_0, \varepsilon_0]$. □

LEMMA 3.3. – *Let ω_ℓ, $\ell \geqslant 0$, be open subsets of $\mathbb{R}^2$ such that $\overline{\omega}_\ell \Subset \omega$ for each ℓ and $\omega = \bigcup_{\ell \geqslant 0} \omega_\ell$. By Lemma 3.2, there exist numbers $\varepsilon_\ell = \varepsilon_\ell(\omega_\ell) > 0$, $\ell \geqslant 0$, such that the symmetric matrices $\mathbf{C}(x) = (g_{ij}(x))$ and $\mathbf{C}^n(x) = (g_{ij}^n(x))$, $n \geqslant 0$ (defined for all $x = (y, x_3) \in \omega \times \mathbb{R}$ as in Lemma 3.2), are positive definite at all points $x = (y, x_3) \in \overline{\Omega}_\ell$, where $\Omega_\ell := \omega_\ell \times]-\varepsilon_\ell, \varepsilon_\ell[$, hence at all points $x = (y, x_3)$ of the open set $\Omega := \bigcup_{\ell \geqslant 0} \Omega_\ell$, which is connected and simply connected.*

Sketch of proof. – The set Ω is connected since it is clearly arcwise connected. To show that Ω is simply connected, let $\boldsymbol{\gamma} \in \mathcal{C}^0([0, 1]; \mathbb{R}^3)$ be a *loop* in Ω. Let the projection operator $\boldsymbol{\pi} : \Omega \to \omega$ be defined by $\boldsymbol{\pi}(y, x_3) = y$ for all $(y, x_3) \in \Omega$. Then the mapping $\widetilde{\boldsymbol{\gamma}} := \boldsymbol{\pi} \circ \boldsymbol{\gamma} \in \mathcal{C}^0([0, 1]; \mathbb{R}^2)$ is a loop in ω, which can be reduced to a point $y^0 \in \omega$ since ω is simply connected. It is then easy to construct a homotopy in Ω that reduces the loop $\boldsymbol{\gamma}$ to the point $(y^0, 0) \in \Omega$. □

LEMMA 3.4. – *The set Ω being defined as in Lemma 3.3, let the functions $R_{qijk} \in \mathcal{C}^0(\Omega)$ and $R_{qijk}^n \in \mathcal{C}^0(\Omega)$, $n \geqslant 0$, be constructed as in Theorem 2.1 from the matrix fields $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $(g_{ij}^n) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, $n \geqslant 0$. Then $R_{qijk} = 0$ in Ω and $R_{qijk}^n = 0$ in Ω for all $n \geqslant 0$.* □

This result, due to Ciarlet and Larsonneur [2], is crucial. Its proof essentially relies on a series of elementary, yet sometimes delicate and lengthy, computations. For details, see *ibid*.

LEMMA 3.5. – *The matrix fields $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $\mathbf{C}^n = (g_{ij}^n) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ defined in Lemma 3.3 satisfy $\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,K} = 0$ for all $K \Subset \Omega$.*

Proof. – Given any compact subset K of Ω, there exists a finite set Λ_K of integers such that $K \subset \bigcup_{\ell \in \Lambda_K} \Omega_\ell$. Since by assumption, $\lim_{n\to\infty} \|a_{\alpha\beta}^n - a_{\alpha\beta}\|_{2,\overline{\omega}_\ell} = 0$ and $\lim_{n\to\infty} \|b_{\alpha\beta}^n - b_{\alpha\beta}\|_{2,\overline{\omega}_\ell} = 0$, $\ell \in \Lambda_K$, it follows that $\lim_{n\to\infty} \|\mathbf{C}_p^n - \mathbf{C}_p\|_{2,\overline{\omega}_\ell} = 0$, $\ell \in \Lambda_k$, $p = 0, 1, 2$, where the matrices $\mathbf{C}_p$ and $\mathbf{C}_p^n$, $n \geqslant 0$, $p = 0, 1, 2$, are defined as in the proof of Lemma 3.2, hence that $\lim_{n\to\infty} \|\mathbf{C}^n - \mathbf{C}\|_{2,\overline{\Omega}_\ell} = 0$, $\ell \in \Lambda_K$. The conclusion then follows from the finiteness of the set Λ_K. □

LEMMA 3.6. – *There exist mappings $\boldsymbol{\theta}^n \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ satisfying*

$$a_{\alpha\beta}^n = \partial_\alpha \boldsymbol{\theta}^n \cdot \partial_\beta \boldsymbol{\theta}^n \quad \textit{and} \quad b_{\alpha\beta}^n = \partial_{\alpha\beta} \boldsymbol{\theta}^n \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta}^n \wedge \partial_2 \boldsymbol{\theta}^n}{|\partial_1 \boldsymbol{\theta}^n \wedge \partial_2 \boldsymbol{\theta}^n|} \right\} \quad \textit{in } \omega, \quad n \geqslant 0,$$

such that $\lim_{n\to\infty} \|\boldsymbol{\theta}^n - \boldsymbol{\theta}\|_{3,\kappa} = 0$ *for all* $\kappa \Subset \omega$.

Sketch of proof. – Given any mapping $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ that satisfies $a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta}$ and $b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \right\}$ in ω, let the mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ be defined by $\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y)$ for all $(y, x_3) \in \Omega$, where $\boldsymbol{a}_3 := \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|}$, and let $g_{ij} := \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$. Then an immediate computation shows that $g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta}$ and $g_{i3} = \delta_{i3}$ in Ω, where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$ and $c_{\alpha\beta} = a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau}$.

In other words, *the matrices (g_{ij}) constructed in this fashion coincide over the set Ω with those defined in* Lemma 3.2. Since Lemmas 3.3, 3.4, and 3.5 together show that all the assumptions of Theorem 2.3 are satisfied by the fields $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ and $\mathbf{C}^n = (g_{ij}^n) \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$, there exist mappings $\boldsymbol{\Theta}^n \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ satisfying $(\boldsymbol{\nabla} \boldsymbol{\Theta}^n)^{\mathrm{T}} \boldsymbol{\nabla} \boldsymbol{\Theta}^n = \mathbf{C}^n$ in Ω, $n \geqslant 0$, such that $\lim_{n\to\infty} \|\boldsymbol{\Theta}^n - \boldsymbol{\Theta}\|_{3,K} = 0$ for all $K \Subset \Omega$. It is then shown that the mappings $\boldsymbol{\theta}^n(\cdot) := \boldsymbol{\Theta}^n(\cdot, 0) \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ indeed satisfy

$$a_{\alpha\beta}^n = \partial_\alpha \boldsymbol{\theta}^n \cdot \partial_\beta \boldsymbol{\theta}^n \quad \text{and} \quad b_{\alpha\beta}^n = \partial_{\alpha\beta} \boldsymbol{\theta}^n \cdot \left\{ \frac{\partial_1 \boldsymbol{\theta}^n \wedge \partial_2 \boldsymbol{\theta}^n}{|\partial_1 \boldsymbol{\theta}^n \wedge \partial_2 \boldsymbol{\theta}^n|} \right\} \quad \text{in } \omega.$$

The relations $\lim_{n\to\infty}\|\boldsymbol{\theta}^n-\boldsymbol{\theta}\|_{3,\kappa}=0$ for all $\kappa\Subset\omega$ follow from the relations $\lim_{n\to\infty}\|\boldsymbol{\Theta}^n-\boldsymbol{\Theta}\|_{3,K}=0$ for all $K\Subset\Omega$, combined with the observations that a compact subset of ω is also one of Ω, that $\boldsymbol{\Theta}(\cdot,0)=\boldsymbol{\theta}$ and $\boldsymbol{\Theta}^n(\cdot,0)=\boldsymbol{\theta}^n$, and finally, that $\|\boldsymbol{\theta}^n-\boldsymbol{\theta}\|_{3,\kappa}\leqslant\|\boldsymbol{\Theta}^n-\boldsymbol{\Theta}\|_{3,\kappa}$. □

Let $(\kappa_i)_{i\geqslant 0}$ be any sequence of subsets of ω that satisfy $\kappa_i\Subset\omega$ and $\kappa_i\subset\operatorname{int}\kappa_{i+1}$ for all $i\geqslant 0$, and $\omega=\bigcup_{i=0}^{\infty}\kappa_i$, and let

$$d_\ell(\boldsymbol{\psi},\boldsymbol{\theta}):=\sum_{i=0}^{\infty}\frac{1}{2^i}\,\frac{\|\boldsymbol{\psi}-\boldsymbol{\theta}\|_{\ell,\kappa_i}}{1+\|\boldsymbol{\psi}-\boldsymbol{\theta}\|_{\ell,\kappa_i}}.$$

Let $\dot{\mathcal{C}}^3(\omega;\mathbf{E}^3):=\mathcal{C}^3(\omega;\mathbf{E}^3)/R$ denote the quotient set of $\mathcal{C}^3(\omega;\mathbf{E}^3)$ by the equivalence relation R, where $(\boldsymbol{\theta},\widetilde{\boldsymbol{\theta}})\in R$ means that there exist a vector $\boldsymbol{a}\in\mathbf{E}^3$ and a matrix $\mathbf{Q}\in\mathbb{O}^3$ such that $\boldsymbol{\theta}(y)=\boldsymbol{a}+\mathbf{Q}\widetilde{\boldsymbol{\theta}}(y)$ for all $y\in\omega$. The set $\dot{\mathcal{C}}^3(\omega;\mathbf{E}^3)$ becomes a *metric space* when it is equipped with the distance $\dot{d}_3$ defined by

$$\dot{d}_3(\dot{\boldsymbol{\theta}},\dot{\boldsymbol{\psi}}):=\inf_{\begin{cases}\boldsymbol{\kappa}\in\dot{\boldsymbol{\theta}}\\ \boldsymbol{\chi}\in\dot{\boldsymbol{\psi}}\end{cases}} d_3(\boldsymbol{\kappa},\boldsymbol{\chi})=\inf_{\begin{cases}\boldsymbol{a}\in\mathbf{E}^3\\ \mathbf{Q}\in\mathbb{O}^3\end{cases}} d_3(\boldsymbol{\theta},\boldsymbol{a}+\mathbf{Q}\boldsymbol{\psi}),$$

where $\dot{\boldsymbol{\theta}}$ denotes the equivalence class of $\boldsymbol{\theta}$ *modulo* R.

The announced continuity of a surface as a function of its two fundamental forms is then a corollary to Theorem 3.1. If d is a metric defined on a set X, the associated metric space is denoted $\{X;d\}$.

THEOREM 3.7. – *Let ω be connected and simply connected open subset of $\mathbb{R}^2$. Let*

$$\begin{aligned}\mathcal{C}_0^2\big(\omega;\mathbb{S}^2_>\times\mathbb{S}^2\big):=\big\{\big((a_{\alpha\beta}),(b_{\alpha\beta})\big)\in\mathcal{C}^2\big(\omega;\mathbb{S}^2_>\big)\times\mathcal{C}^2\big(\omega;\mathbb{S}^2\big);&\\ \partial_\beta C_{\alpha\sigma\tau}-\partial_\sigma C_{\alpha\beta\tau}+C^\mu_{\alpha\beta}C_{\sigma\tau\mu}-C^\mu_{\alpha\sigma}C_{\beta\tau\mu}=b_{\alpha\sigma}b_{\beta\tau}-b_{\alpha\beta}b_{\sigma\tau}\ \textit{in}\ \omega,&\\ \partial_\beta b_{\alpha\sigma}-\partial_\sigma b_{\alpha\beta}+C^\mu_{\alpha\sigma}b_{\beta\mu}-C^\mu_{\alpha\beta}b_{\sigma\mu}=0\ \textit{in}\ \omega\big\}.&\end{aligned}$$

Given any element $((a_{\alpha\beta}),(b_{\alpha\beta}))\in\mathcal{C}_0^2(\omega;\mathbb{S}^2_>\times\mathbb{S}^2)$, let $F(((a_{\alpha\beta}),(b_{\alpha\beta})))\in\dot{\mathcal{C}}^3(\omega;\mathbf{E}^3)$ denote the equivalence class modulo R *of any $\boldsymbol{\theta}\in\mathcal{C}^3(\omega;\mathbf{E}^3)$ that satisfies*

$$a_{\alpha\beta}=\partial_\alpha\boldsymbol{\theta}\cdot\partial_\beta\boldsymbol{\theta}\quad\textit{and}\quad b_{\alpha\beta}=\partial_{\alpha\beta}\boldsymbol{\theta}\cdot\left\{\frac{\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}|}\right\}\quad\textit{in}\ \omega.$$

Then the mapping

$$F:\big\{\mathcal{C}_0^2\big(\omega;\mathbb{S}^2_>\times\mathbb{S}^2\big);d_2\big\}\to\big\{\dot{\mathcal{C}}^3\big(\omega;\mathbf{E}^3\big);\dot{d}^3\big\}$$

defined in this fashion is continuous. □

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