

On the nonlinear type singularities for semilinear Cauchy problems

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Abstract

We consider the Cauchy problem for the semilinear wave equation. The Cauchy data are assumed to be conormal with respect to a point, and the source term is polynomial with respect to the solution and its first derivatives. Thanks to the study of multiplicative properties of some refined hyperbolic conormal spaces, we improve the known results about the nonlinear type singularities of the solution. *To cite this article: D. Fang et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 453–458.*

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Sur les singularités de type non-linéaire pour des problèmes de Cauchy semi-linéaires

Résumé

On considère le problème de Cauchy pour l'équation des ondes semi-linéaire à données de Cauchy conormales par rapport à un point, et à terme source polynomial par rapport à la solution et à ses dérivées premières. Grâce à l'étude des propriétés multiplicatives d'espaces conormaux hyperboliques précisés, on améliore les résultats connus sur la taille des singularités de type non-linéaire de la solution. *Pour citer cet article : D. Fang et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 453–458.*

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On considère le problème de Cauchy

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} = f(t, x, u, \nabla u), \\ u|_{t=0} = g_0, \quad \partial_t u|_{t=0} = g_1 \end{cases} \quad (1)$$

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où $t \in \mathbb{R}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, f est une fonction C^∞ de ses arguments, et où les donnés de Cauchy g_j ($j = 0, 1$) sont conormales d'indice $s-j$ par rapport à $\{0\}$, avec $s > (n+1)/2 + 1$:

$$g_j \in I^{-s-n/4+j}(\mathbb{R}^n, \{O\}), \quad j = 0, 1 \quad (2)$$

voir Hörmander [9]. Il nous sera commode d'utiliser la notation :

$$I^{-s-n/4+j}(\mathbb{R}^n, \{O\}) = {}^\infty H^{s-j}(\mathbb{R}^n, \{O\}). \quad (3)$$

Bony [3] a prouvé que

$$u \in {}^\infty H^s(\mathbb{R}^{n+1}, \Gamma, \{O\}) = I^{-s-(n+1)/4}(\mathbb{R}^{n+1}, \Gamma, \{O\}), \quad (4)$$

où cet espace est défini comme dans [9], en considérant les champs de vecteurs tangents à $\Gamma \setminus \{O\}$ et $\{O\}$, Γ désignant le cône de lumière

$$\Gamma = \{(t, x) \in \mathbb{R}^{n+1} \mid t^2 - |x|^2 = 0\}. \quad (5)$$

Dans le fibré cotangent à \mathbb{R}^{n+1} privé de sa section nulle, soient Λ_0 le fibré conormal à $\{O\}$ et Λ_1 l'adhérence du fibré conormal à $\Gamma \setminus \{O\}$:

$$\Lambda_0 = \{(t, x; \tau, \xi) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus O) \mid t = x = 0\}, \quad (6)$$

$$\Lambda_1 = \{(t, x; \tau, \xi) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus O) \mid t^2 - |x|^2 = 0, t\xi + \tau x = 0\}. \quad (7)$$

Le résultat de Bony montre que $\text{WF}(u) \subset \Lambda_0 \cup \Lambda_1$, que des singularités de type «non-linéaire» sont susceptibles d'apparaître dans $\Lambda_0 \setminus \Lambda_1$, et que microlocalement dans $\Lambda_0 \setminus \Lambda_1$, u est dans ${}^\infty H^s(\mathbb{R}^{n+1}, \{O\})$. Le but de ce travail est d'améliorer l'estimation de ces singularités. On établit le Théorème :

THÉORÈME. — *On suppose que f est polynomial en $(u, \nabla u)$ et que g_j satisfait (2). Alors, microlocalement dans $\Lambda_0 \setminus \Lambda_1$, la solution u de (1) est dans ${}^\infty H^{s_0}(\mathbb{R}^{n+1}, \{O\})$, avec $s_0 = 2s - \frac{n}{2} + \frac{1}{2}$. Dans le cas faiblement semilinéaire (c.a.d quand f ne dépend pas de ∇u), avec $s > (n+1)/2$, on peut prendre $s_0 = 2s - \frac{n}{2} + \frac{5}{2}$.*

Notre preuve est basée sur l'utilisation d'espaces conormaux hyperboliques $J_c^{\mu, \nu}$, dont les éléments sont, microlocalement dans $\Lambda_0 \setminus \Lambda_1$, dans ${}^\infty H^{s'}(\mathbb{R}^{n+1}, \{O\})$ avec $s' = -(\mu + \nu) - (n+1)/2$, et dont les symboles dégénèrent le long du cône caractéristique

$$C = \{(\tau, \xi) \in \mathbb{R}^{n+1} \setminus O \mid \tau^2 - |\xi|^2 = 0\}.$$

En dehors de $\{t = x = 0\}$, ils sont dans ${}^\infty H^s(\mathbb{R}^{n+1}, \{O\}, \Gamma \setminus \{O\})$ avec $s = -\mu - \frac{n}{2}$. En utilisant des propriétés multiplicatives de ces espaces, que nous établissons à l'aide de techniques inspirées de Foschi, Klainerman [8], nous montrons que $u \in J_c^{\mu, \mu+n-1}$ (et $u \in J_c^{\mu, \mu+n-3}$ dans le cas faiblement semi-linéaire), ce qui implique le Théorème.

1. Introduction

We consider the semilinear Cauchy problem (1), where f is a smooth function of its arguments, and conormal Cauchy data g_j ($j = 0, 1$) with respect to $\{O\}$, Besov index $s - j$, $s > (n+1)/2 + 1$. See (2), Hörmander [9], and our notation (3). Bony proved [3] that u satisfies (4), the last space being defined like in

[9] using vector fields tangent to $\Gamma \setminus \{O\}$ and $\{O\}$, where Γ is the light cone (5). Bony's result implies that $\text{WF}(u) \subset \Lambda_0 \cup \Lambda_1$, where Λ_0, Λ_1 are defined in (6), (7), and that “nonlinear” type singularities belonging to ${}^\infty H^s(\mathbb{R}^{n+1}, \{O\})$ may appear microlocally in $\Lambda_0 \setminus \Lambda_1$. Our goal is to improve this result. We prove the following theorem:

THEOREM 1. – Assume that f is polynomial with respect to $(u, \nabla u)$ and that g_j satisfies (2). Then, microlocally in $\Lambda_0 \setminus \Lambda_1$, the solution u to (1) belongs to ${}^\infty H^{s_0}(\mathbb{R}^{n+1}, \{O\})$, with $s_0 = 2s - \frac{n}{2} + \frac{1}{2}$. In the weakly nonlinear case (that is when f does not depend on ∇u), one can take $s_0 = 2s - \frac{n}{2} + \frac{5}{2}$.

Our proof uses some hyperbolic conormal spaces.

2. Hyperbolic conormal spaces

Definition 1. – For $\mu, \nu \in \mathbb{R}$, $S^{\mu, \nu}$ is the space of symbols $b = b(t, x, \tilde{\tau}, \tilde{\xi}) \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ such that

$$\partial_{t,x}^\beta \partial_{\tilde{\tau}}^k \partial_{\tilde{\xi}}^\alpha b = O((\tilde{\tau}, \tilde{\xi})^{\mu - |\alpha|} (\tilde{\tau})^{\nu - k}), \quad (8)$$

where $\langle \theta \rangle = 1 + |\theta|$. $S_c^{\mu, \nu}$ is the space of symbols $a = a(t, x, \tau, \xi) \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ such that: Outside a conical neighborhood of $C^\pm = \{\tau \mp |\xi| = 0\}$, $a \in S^{\mu+\nu}$ (ordinary symbol). In a (small) conical neighborhood of C^\pm , setting $\tilde{\tau} = \tau \mp |\xi|$, $\tilde{\xi} = \xi$, then $a(t, x, \tau, \xi) = b(t, x, \tilde{\tau}, \tilde{\xi})$ with $b \in S^{\mu, \nu}$ for large $\tilde{\xi}$. We set $S_c^{\mu, \nu} = S_c^{\mu, \nu} + S_c^{\mu, \nu}$.

Definition 2. – For $\mu, \nu \in \mathbb{R}$, $J_c^{\mu, \nu}$ is the space of distributions $u \in \mathcal{D}'(\mathbb{R}^{n+1})$ such that

$$u(t, x) = \int e^{i(t\tau + x\xi)} a(t, x, \tau, \xi) d\tau d\xi, \quad a \in S_c^{\mu, \nu}.$$

We set $J_c^{\mu, \nu} = J_c^{\mu, \nu} + J_c^{\mu, \nu}$.

We have the following properties

$$u \in J_c^{\mu, \nu} \cap \mathcal{E}' \rightarrow \hat{u} \in S_c^{\mu, \nu}, \quad (9)$$

$$u \in J_c^{\mu, \nu} \rightarrow \text{WF}(u) \subset \Lambda_0 \cup \Lambda_1, \quad (10)$$

$$J_c^{\mu, -\infty} = I^{\mu+(n-1)/4}(\mathbb{R}^{n+1}, \Lambda_1). \quad (11)$$

Note that the last space is the natural space of solutions of (1) in the linear case, with $\mu = -s - \frac{n}{2}$. After restriction to $\{(t, x) \neq 0\}$, then

$$J_c^{\mu, \nu} = I^{k+(n-1)/4}(\mathbb{R}^{n+1}, \Lambda_1) = {}^\infty H^s(\mathbb{R}^{n+1} \setminus \{O\}, \Gamma \setminus \{O\}), \quad (12)$$

with $s = -\mu - \frac{n}{2}$. Microlocally in $\Lambda_0 \setminus \Lambda_1$, then

$$J_c^{\mu, \nu} = I^{\mu+\nu+(n-1)/4}(\mathbb{R}^{n+1}, \{O\}) = {}^\infty H^{s'}(\mathbb{R}^{n+1}, \{O\}), \quad (13)$$

with $s' = -(\mu + \nu) - (n + 1)/2$.

$$\text{If } u \in J_c^{\mu, \nu} \text{ and } \square u \in J_c^{\mu+1, \nu}, \text{ then } u \in J_c^{\mu, \nu-1}. \quad (14)$$

With the notation (4) we have, for any $\varepsilon > 0$,

$${}^\infty H^s(\mathbb{R}^{n+1}, \Gamma, \{O\}) \subset J_c^{\mu, -1/2} \subset {}^\infty H^{s-\varepsilon}(\mathbb{R}^{n+1}, \Gamma, \{O\}), \quad (15)$$

where $\mu = -s - \frac{n}{2}$.

The stability under multiplication is obviously crucial for our study. We have the following result:

THEOREM 2. – Let $u_1, u_2 \in J_c^{\mu, \nu}$. Then $u_1 u_2 \in J_c^{\mu, \nu}$ when $\mu + n - 1 \leq \nu < -1$.

Because ${}^\infty H^t(\mathbb{R}^{n+1}, \Gamma, \{O\})$ is also stable under multiplication when $t > (n+1)/2$, from (15), (14) and Theorem 2 we obtain:

COROLLARY 3. – When f is polynomial with respect to $(u, \nabla u)$, the solution u to (1) belongs to $J_c^{\mu, \mu+n-1}$ (and to $J_c^{\mu, \mu+n-3}$ in the weakly semilinear case).

Now Theorem 1 follows by using (13).

3. Sketch of the proof of the multiplicative property

We consider $v = u_1 u_2$ with $\hat{u}_j \in S_c^{\mu, \nu}$, and we want to prove that $\hat{v} \in S_c^{\mu, \nu}$. From the stability under convolution of $S^{\mu+\nu}$ when $\mu + \nu < -n - 1$, the two essential steps are the following:

Step 1: A ++ case

We assume that $\hat{u}_j \in S_{c+}^{\mu, \nu}$ is supported in $\{\varepsilon|\xi| \leq \tau \leq C|\xi|\}$, where $0 < \varepsilon < C$. After a first integration taking advantage of the stability of $S^\nu(\mathbb{R})$ under convolution if $\nu < -1$, we get

$$\hat{v}(\tau, \xi) = c \int g(\tau - |\eta| - |\xi - \eta|, \eta, \xi - \eta) d\eta, \quad (16)$$

where $g = g(\lambda, \eta, \theta)$ is a product type symbol with degrees ν, μ, μ , and c is a constant whose value may change from a line to another. For fixed $\xi \in \mathbb{R}^n \setminus O$, we transform (16) like in Foschi, Klainerman [8], using the family of ellipsoids $E_\rho = \{\eta \mid |\eta| + |\xi - \eta| = \rho\}$,

$$\begin{aligned} \hat{v}(\tau, \xi) &= c \int_{\rho \geq |\xi|} (\rho^2 - |\xi|^2)^{(n-3)/2} d\rho \\ &\quad \times \int g(\tau - \rho, \eta_0, \xi - \eta_0) (\rho^2 - s^2 |\xi|^2) (1 - s^2)^{(n-3)/2} ds d\omega', \end{aligned} \quad (17)$$

where the last integration is performed for $-1 \leq s \leq 1$, $\omega' \in S_{n-1} \cap \xi^\perp$, $\eta_0 = \frac{1}{2}(\rho^2 - |\xi|^2)^{1/2} \cdot (1 - s^2)^{1/2} \omega' + (s\rho + |\xi|) \frac{\xi}{|\xi|}$.

We have $|\eta_0| = (\rho + s|\xi|)/2$, $\xi - \eta_0 = (\rho - s|\xi|)/2$, therefore

$$\begin{aligned} |\hat{v}(\tau, \xi)| &\leq c \int_{\rho \geq |\xi|} (\rho^2 - |\xi|^2)^{(n-3)/2} \langle \tau - \rho \rangle^\nu d\rho \\ &\quad \times \int_0^1 \langle \rho + s|\xi| \rangle^\mu \langle \rho - s|\xi| \rangle^\mu (\rho^2 - s^2 |\xi|^2) (1 - s^2)^{(n-3)/2} ds. \end{aligned} \quad (18)$$

When $\tau \geq M|\xi|$ and $M > C$, we get $\rho \pm s|\xi| \simeq \rho \simeq \tau$, hence, for large τ :

$$|\hat{v}(\tau, \xi)| \leq c \tau^{2\mu+n-1} \leq c \tau^{\mu+\nu} \quad \text{if } \nu \geq \mu + n - 1. \quad (19)$$

The derivatives of \hat{v} are estimated in the same way. Note however that $\partial_\tau^\alpha \hat{v}(\tau, \xi)$ corresponds to (17) with $g(\lambda, \eta, \theta)$ replaced by $\partial_\lambda^\alpha g(\lambda, \eta, \theta)$, and ν by $\nu - \alpha$. But in (19) the condition $\nu \geq \mu + n - 1$ may be wrong if ν is replaced by $\nu - \alpha$; in fact, the wanted estimate $|\partial_\tau^\alpha \hat{v}(\tau, \xi)| \leq C \tau^{\mu+\nu-\alpha}$ is obtained thanks to integration by parts with respect to the ρ variable, using the symmetry of τ and ρ in $(\partial_\tau^\alpha g)(\tau - \rho, \eta_0, \xi - \eta_0)$.

When $\tau \leq M|\xi|$, setting $1 - s = t$ and $\rho - |\xi| = \tilde{\rho}$ in (18) and using Lemma 4.2 of [8], we get, for large $|\xi|$:

$$|\hat{v}(\tau, \xi)| \leq c|\xi|^\mu \int_0^{+\infty} \langle \tau - |\xi| - \tilde{\rho} \rangle^v \langle \tilde{\rho} \rangle^{\mu+(n+1)/2} \tilde{\rho}^{(n-3)/2} d\tilde{\rho} \quad (20)$$

which gives immediately

$$|\hat{v}(\tau, \xi)| \leq c|\xi|^\mu \langle \tau - |\xi| \rangle^v, \quad \text{if } \tau - |\xi| \leq 0. \quad (21)$$

If $\tau - |\xi| > 0$, decomposing the $\tilde{\rho}$ integral in (20) along $]0, \frac{1}{2}(\tau - |\xi|)[$, $]\frac{1}{2}(\tau - |\xi|), \frac{3}{2}(\tau - |\xi|)[$ and $]\frac{3}{2}(\tau - |\xi|), +\infty[$, and using again Lemma 4.2 of [8], we obtain three terms respectively bounded by $c|\xi|^\mu \langle \tau - |\xi| \rangle^v$, $c|\xi|^\mu \langle \tau - |\xi| \rangle^{\mu+n-1}$ and $c|\xi|^\mu \langle \tau - |\xi| \rangle^{v+\mu+n}$, which imply the expected estimate $|\hat{v}(\tau, \xi)| \leq c \langle \tau, \xi \rangle^\mu \langle \tau - |\xi| \rangle^v$ thanks to $v \geq \mu + n - 1$ and $\mu + n \leq 0$.

The derivatives with respect to the new variables $\tilde{\tau}, \tilde{\xi}$ given in (8) are estimated similarly, using again integration by parts with respect to the $\tilde{\rho}$ variable in the case of the $\tilde{\tau}$ derivatives when $\frac{1}{2}(\tau - |\xi|) \leq \tilde{\rho} \leq \frac{3}{2}(\tau - |\xi|)$.

Step 2: A \pm case

We assume that $\hat{u}_1 \in S_{c^-}^{\mu, v}$ and $\hat{u}_2 \in S_{c^+}^{\mu, v}$ are supported in $\{|\tau| \leq C|\xi|\}$. The method is similar, using now the family of hyperboloids $\mathcal{H}_\rho = \{\eta \mid |\eta| - |\xi - \eta| = \rho\}, -|\xi| \leq \rho \leq |\xi|$.

Remark 1. – In Theorem 2 the condition $\mu + n - 1 \leq v$ is optimal. Indeed, consider, for instance,

$$u_j(t, x) = \int e^{i(x\xi + t|\xi|)} a_j(t, x, \xi) d\xi,$$

where $a_j \in S^\mu$, compactly supported with respect to (t, x) , $j = 1, 2$. Then $u_j \in J_{c^+}^{\mu, -\infty}$, and, using the stationary phase theorem, we can prove that $v = u_1 u_2$ satisfies $\hat{v} \in S^{2\mu+n-1}$ for $\tau > |\xi|$ with principal part given by

$$(2\pi)^{n+1} |(\tau, \xi)|^{n-1} \int_{S_{n-1}} F(\tau, \xi, \omega) a(0, 0; \lambda\omega) b(0, 0; \lambda\omega) d\omega,$$

where $\lambda = \frac{1}{2} \frac{\tau^2 - |\xi|^2}{\tau - \xi \cdot \omega}$, $F(\tau, \xi, \omega) = (1 - \frac{\xi \cdot \omega - \lambda}{|\xi - \lambda\omega|})^{-1}$. We have

$$\begin{aligned} \frac{1}{2}(\tau - |\xi|) &\leq \lambda \leq \frac{1}{2}(\tau + |\xi|), \\ \frac{1}{2}(\tau - |\xi|) &\leq |\xi - \lambda\omega| \leq \frac{1}{2}(\tau + |\xi|), \\ \frac{1}{2} &\leq F(\tau, \xi, \omega) \leq \frac{1}{2} \frac{\tau + |\xi|}{\tau - |\xi|}, \end{aligned}$$

thus (13) shows that the value $v = \mu + n - 1$ is in general the best possible for the multiplicative property.

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