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Conformal fields, restriction properties, degenerate representations and SLE

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Abstract We relate the Schramm–Loewner Evolution processes (SLE) to highest-weight representations of the Virasoro Algebra. The restriction properties of SLE that have been recently derived in [19] play a crucial role. In this setup, various considerations from conformal field theory can be interpreted and reformulated via SLE. This enables one to make a concrete link between the two-dimensional discrete critical systems from statistical physics and conformal field theory. ***To cite this article: R. Friedrich, W. Werner, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 947–952.***

Champs conformes, propriété de restriction, représentations dégénérées et SLE

Résumé Nous relions le processus de Schramm–Loewner (SLE) à certaines représentations de plus haut poids dégénérées de l'algèbre de Virasoro. Les propriétés de restriction du SLE étudiées dans [19] s'avèrent être importantes pour établir ce lien. Par ailleurs, diverses considérations et relations de la théorie conforme des champs peuvent ainsi être interprétées en termes du SLE. Ceci permet de faire le lien entre les modèles issus de la physique statistique et la théorie conforme des champs. ***Pour citer cet article : R. Friedrich, W. Werner, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 947–952.***

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Il a été récemment [24,17] prouvé que les processus de Schramm–Loewner (SLE) introduits par Schramm dans [23] sont la limite d'échelle de plusieurs systèmes bidimensionnels issus de la physique statistique. En fait, ce sont les seules limites invariantes conformes possibles d'une large classe de modèles (percolation critique, marches auto-évitantes, Ising, Potts) pour lesquels les physiciens théoriciens ont développé la théorie conforme des champs (dont le lien avec ces modèles est mathématiquement encore mal compris) qui leur a permis de prédire la valeur des exposants critiques qui décrivent ces systèmes (voir par exemple la compilation d'articles dans [7]). Dans le formalisme de la théorie conforme des champs, ces

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exposants apparaissent en termes de représentations de plus haut poids dégénérées de l'algèbre de Virasoro (voir par exemple les articles dans les recueils [7,6]).

L'étude directe du SLE a permis de prouver mathématiquement (modulo l'invariance conforme) la valeur de ces exposants [13–16], et suggère donc l'existence d'un lien direct entre les représentations de l'algèbre de Virasoro et le SLE ; Ce lien permet de clarifier la relation entre les systèmes discrets et les champs associés.

En exploitant la récente étude de la propriété de restriction du SLE dévelopée dans [19], nous établissons ce lien. Pour faciliter l'exposition, nous nous limitons ici au cas le plus simple, à savoir le cas du comportement au bord du $\mathrm{SLE}_{8/3}$ chordal (correspondant dans le formalisme algébrique à une charge centrale nulle) et décrirons le cas général dans un article plus complet à venir. Nous définissons les quantités

$$B_n(x_1,\ldots,x_n)=\lim_{\varepsilon\to 0}\varepsilon^{-2n}\mathbf{P}\left[\bigcap_{j=1}^{n}\left\{\gamma\cap[x_j,x_j+\mathrm{i}\varepsilon\sqrt{2}]\neq\emptyset\right\}\right]$$

et montrons qu'elles vérifient des équations (identité de Ward par exemple) prédites pour les fonctions de corrélation de la théorie de champs, comme par exemple :

$$B_{n+1}(x,x_1,\ldots,x_n)=\left[\frac{\alpha}{x^2}-\sum_{j=1}^{n}\left\{\left(\frac{1}{x_j-x}+\frac{1}{x}\right)\partial_{x_j}-\frac{2}{(x_j-x)^2}\right\}\right]B_n(x_1,\ldots,x_n)$$

et

$$-4\left(\sum_{j=1}^{n}\frac{1}{x_j^2}\right)B_n+\left(\sum_{j=1}^{n}\frac{2}{x_j}\partial_{x_j}\right)B_n+\frac{\kappa}{2}\left(\sum_{j=1}^{n}\partial_{x_j}\right)^2B_n=0.$$

À partir de ces relations, nous décrivons ensuite une représentation de plus haut poids dégénérée de l'algèbre de Lie $\mathcal{A}$ des champs de vecteurs sur le cercle. Ceci permet d'identifier les valeurs $\kappa=8/3$ et $\alpha=5/8$ à partir de la propriété de restriction. Ce type de considérations peut être généralisé à d'autres SLE_κ et permet de faire le lien avec les représentations de l'algèbre de Virasoro (l'extension centrale de $\mathcal{A}$).

1. Introduction

Conformal field theory has been remarkably successful in predicting the critical behaviour of two-dimensional systems from statistical physics (see [3,4] and for instance the compilation of papers in [7]). One of the starting points [21,3] for this theory is that each system should be (in its scaling limit) corresponding to a conformal field (these fields are classified according to their central charge). The behaviour of the system should then be described by critical exponents which physicists identify as highest-weights of certain degenerate representations of infinite-dimensional Lie Algebras (this motivated also many mathematical papers on representation theory, one should probably cite here at least all the papers reprinted in [6], for the representation theory of the Virasoro Algebra and background, see [9,8]). This applies for instance to Ising and Potts models, percolation, self-avoiding walks. However, as for instance pointed out in [11], many fundamental features have remained unclear at least for mathematicians. For instance, the actual relation between the discrete system and the fields, i.e., the meaning of the field in terms of the discrete system, the interpretation of the equations leading to the highest-weight representation were rather mysterious.

In 1999, Oded Schramm [23] defined a one-parameter family of random curves based on Loewner's differential equation, SLE_κ indexed by the positive real parameter κ (SLE stands for Stochastic- or Schramm–Loewner Evolution). These random curves are the only ones which combine conformal invariance and a Markovian-type property. Provided that the scaling limit of interfaces in the above-mentioned models exist and are conformally invariant, then the limiting objects must therefore be one of the SLE_κ curves. This has now been rigourously proved in some cases (critical site percolation on the triangular lattice [24] and uniform spanning trees [17]). For a general discussion of the conjectured relation between the discrete models and SLE, see [22]. In this SLE setting, the critical exponents are simply principal eigenvalues of some differential operators [13–16]. This led to a complete mathematical proof for the value of critical exponents for those models that have been proved to be conformally invariant (see, e.g., [16,25]). In order to confirm rigorously the conjectures for the other models, the missing step is to derive their conformal invariance.

It is therefore natural to think that SLE should be related to conformal field theory (as for instance recognized in [1]) and to highest-weight representations of the Virasoro Algebra. Our goal in the present Note is to explain that this is indeed the case. In order to make this explantion as clear and rigorous as possible (and also to keep this Note short), we will restrict ourselves to the simplest case: the "boundary behaviour" of $\mathrm{SLE}_{8/3}$ that corresponds in the field theory language to a zero central charge and conjecturally corresponds to the scaling limit of the half-plane self-avoiding walk [18]. The other cases (non-zero central charge, behaviour in the bulk) will be detailed in forthcoming papers.

2. SLE facts

The chordal SLE_κ curve γ is characterized as follows: the conformal maps g_t from $\mathbb{H} \setminus \gamma[0,t]$ onto $\mathbb{H}$ such that $g_t(z) = z + \mathrm{o}(1)$ when $z \to \infty$ solve the ordinary differential equation $\partial_t g_t(s) = 2/(g_t(z) - W_t)$ (and is started from $g_0(z) = z$), where $W_t = \sqrt{\kappa}\beta_t$ (here and in the sequel, $(\beta_t,\ t \geqslant 0)$ is a standard real-valued Brownian motion with $\beta_0 = 0$). In other words, γ_t is precisely the point such that $g_t(\gamma_t) = W_t$. See, e.g., [13,22] for the definition and properties of SLE, or [12,26] for reviews.

In the recent paper [19], it is shown how to measure the distortion of the law of an SLE when its image is mapped conformally from a subdomain onto some other domain. In particular, for the special value $\kappa = 8/3$, the SLE curve γ (which is conjectured to be the scaling limit of a certain measure on self-avoiding curves, see [18] for a discussion of this conjecture and some of its consequences – this conjecture is conforted by numerical simulations [10]) has the *conformal restriction property* that we now briefly describe:

Suppose that H is a simply connected open subset of the upper-half plane $\mathbb{H}$ such that $\mathbb{H} \setminus H$ is bounded and bounded away from 0. Assume also that γ is chordal $\mathrm{SLE}_{8/3}$. Then, the law of γ conditioned to remain in H is identical to the law of $\Phi(\gamma)$ where Φ is the conformal map from $\mathbb{H}$ onto H that fixes the two boundary points 0 and ∞ and $\Phi(z) \sim z$ as $z \to \infty$. Actually [19], $\mathrm{SLE}_{8/3}$ is the unique simple random curve with this property. As argued towards the begining of the paper [19], if γ satisfies the restriction property, then there exists $\alpha > 0$ such that

$$P[\gamma \subset H] = \Phi'(0)^{-\alpha}. \tag{1}$$

It is proved in [19] that the value of α corresponding to $\mathrm{SLE}_{8/3}$ is $5/8$ (but we will not use this fact in this Note, as one of our goals is to explain why one can recover it from algebraic considerations as was for instance predicted in [5]).

3. Boundary behaviour

When $\kappa < 8$ and $\varepsilon \to 0$, the probability that chordal SLE_κ intersects the ε-neighbourhood of the real point $x \neq 0$ can be shown to decay (up to a multiplicative constant) like ε^s when $\varepsilon \to 0$, where $s = (8/\kappa) - 1$. More generally, one defines for $x_1, \dots, x_n \in \mathbb{R} \setminus \{0\}$,

$$B_n(x_1,\dots,x_n) = \lim_{\varepsilon_1,\dots,\varepsilon_n \to 0} \varepsilon_1^{-s} \dots \varepsilon_n^{-s} \mathbf{P}\left[\bigcap_{j=1}^n \left\{\gamma \cap [x_j, x_j + \mathrm{i}\varepsilon_j\sqrt{2}] \neq \emptyset\right\}\right]$$

(the choice of these vertical slits enables one to have simple renormalizing constants later on). It is easy to see using (1) that if the restriction property holds, then $s = 2$ and $B_1(x) = \alpha/x^2$. The functions B_n corresponding to correlation functions in conformal field theory. They have the following properties:

PROPOSITION 3.1. –
(1) *Ward-type identity*: *if the restriction property holds* (*with the exponent* α), *then for all* $n \geqslant 1$,

$$B_{n+1}(x, x_1, \ldots, x_n) = \left[\frac{\alpha}{x^2} - \sum_{j=1}^{n}\left\{\left(\frac{1}{x_j - x} + \frac{1}{x}\right)\partial_{x_j} - \frac{2}{(x_j - x)^2}\right\}\right] B_n(x_1, \ldots, x_n). \tag{2}$$

(2) *Evolution equation*:

$$-2s\left(\sum_{j=1}^{n}\frac{1}{x_j^2}\right)B_n + \left(\sum_{j=1}^{n}\frac{2}{x_j}\partial_{x_j}\right)B_n + \frac{\kappa}{2}\left(\sum_{j=1}^{n}\partial_{x_j}\right)^2 B_n = 0. \tag{3}$$

Proof (*sketch*). – The first identity is in fact a direct consequence of the restriction property: suppose that the real numbers $x_1, \ldots, x_n$ are fixed and let us focus on the event that γ does intersect all the slits $[x_j, x_j + \mathrm{i}\varepsilon\sqrt{2}]$. Let us also choose another point $x \in \mathbb{R}$ and a small δ. Now, either γ avoids $[x, x + \mathrm{i}\delta\sqrt{2}]$ or it does also hit it. The probability of the first event can be expressed using the restriction property, while the second one involves the $(n + 1)$-points functions B_{n+1}. The sum of the two probabilities remains independent of δ. Looking at the δ^2 term in its expansion yields (2). Let us stress that these identities together with the knowledge of $B_1(x_1) = \alpha/x_1^2$ fully determine all the functions B_n. One can also write $B_0 = 1$ as a function of 0 variables.

The Markovian property of the SLE curves (which is a straightforward consequence of the stationarity of the increments of the Brownian motion β) goes as follows: the law of $\gamma[t, \infty)$ given $\gamma[0, t]$ is the image of an independent copy $\tilde{\gamma}$ of γ under a conformal map from $\mathbb{H}$ on $\mathbb{H} \setminus \gamma[0, t]$ which maps 0 onto $\gamma(t)$ and ∞ onto itself. In other words, the conditional law of $g_t(\gamma[t, \infty)) - W_t$ is independent of $\gamma[0, t]$ and identical to that of the initial SLE. This shows immediately that for all fixed reals $x_1, \ldots, x_n$, the processes

$$\left|g_t'(x_1)\right|^s \cdots \left|g_t'(x_n)\right|^s B_n\left(g_t(x_1) - W_t, \ldots, g_t(x_n) - W_t\right)$$

are local martingales. Itô's formula then yields immediately (3). □

4. Representations

It is interesting to focus on the asymptotic expansion of $B_{n+1}(x, x_1, \ldots, x_n)$ when $x \to 0$. It is natural to define the operators

$$\mathcal{L}_{-N} = \sum_{j}\left\{-x_j^{1-N}\partial_j + 2(N-1)x_j^{-N}\right\}.$$

Note that these operators satisfy the commutation relation

$$[\mathcal{L}_N, \mathcal{L}_M] = (N - M)\mathcal{L}_{N+M}. \tag{4}$$

The Algebra generated by vectors satisfying this commutation relation will be denoted by $\mathcal{A}$ and is often viewed as the Lie algebra of vector fields of the circle, i.e., the algebra generated by $e_N = -z^{N+1}\mathrm{d}/\mathrm{d}z$, $N \in \mathbb{Z}$ (which satisfy the same commutation relation). One can rewrite the previous Proposition using the operators $\mathcal{L}$: the Ward-type identity becomes

$$B_{n+1}(x, x_1, \ldots, x_n) = \frac{\alpha}{x^2}B_n(x_1, \ldots, x_n) + \sum_{N \geqslant 1} x^{N-2}\mathcal{L}_{-N}B_n(x_1, \ldots, x_n) \tag{5}$$

if $|x| < \min(x_1, \ldots, x_n)$. The evolution equation becomes (if $s = 2$, i.e., if the restriction property holds)

$$\left(\frac{\kappa}{2}\mathcal{L}_{-1}^2 - 2\mathcal{L}_{-2}\right)B_n = 0. \tag{6}$$

It is easy to make a little computation to see that $\kappa = 8/3$ and $\alpha = 5/8$ if the previous proposition holds for some family of functions B_n. This computation has some similarities with the computation that shows that a highest-weight representation of $\mathcal{A}$ that is degenerate at level two has a highest weight equal to $5/8$. We now show that this is not just a similarity. Note that scaling implies that $\mathcal{L}_0 B_n = 0$, so that the representation is not simply given in terms of the $\mathcal{L}$'s.

Here is one way to construct a highest-weight representation of $\mathcal{A}$. Define the vector $B = (B_0, B_1, \ldots)$. Suppose that $w = (w_0, w_1, \ldots)$ is such that w_n is a (rational) function of the n variables $x_1, \ldots, x_n$. We define the operators l_N is such a way that

$$w_{n+1}(x, x_1, \ldots, x_n) = \sum_{N \in \mathbb{Z}} x^{N-2}\big(l_{-N}(w)\big)_n(x_1, \ldots, x_n) \tag{7}$$

when $x \to \infty$. In other terms the function of n variables in $l_{-N}(w)$ is the x^{N-2} term in the Laurent expansion of $w_{n+1}(x, x_1, \ldots, x_n)$. Eq. (5) shows that

$$l_N(B) = \begin{cases} (0, 0, \ldots) & \text{if } N > 0, \\ (\alpha B_0, \alpha B_1, \ldots) & \text{if } N = 0, \\ (\mathcal{L}_N B_0, \mathcal{L}_N B_1, \ldots) & \text{if } N < 0. \end{cases} \tag{8}$$

We then define the vector space V generated by B and all the vectors $l_{N_1} \cdots l_{N_r}(B)$ for $N_1 \leqslant \cdots \leqslant N_r < 0$. It is then possible to prove (without using the evolution equation) that:

PROPOSITION 4.1. –
- *For $N_1 \leqslant \cdots \leqslant N_r < 0$, the function of n variables in $l_{N_1} \cdots l_{N_r}(B)$ is $\mathcal{L}_{N_1} \cdots \mathcal{L}_{N_r} B_n$.*
- *V is stable under all l_N's (for all $N \in \mathbb{Z}$, not only for $N \leqslant 0$).*
- *$[l_M, l_N]w = (M - N)l_{M+N}w$ for all $w \in V$ and all $N, M \in \mathbb{Z}$ (note again that M and N can be positive).*

Proof (*sketch*). – The first statement can be proved inductively using (5). It implies immediately the third statement for all negative N, M. In order to obtain the commutation relation for all N, M, and due to the fact that $l_N(B) = 0$ for all positive N, it in fact suffices to check it if only M is positive. This can then be done again thanks to (5) and the first statement. Finally, note that the second statement is a consequence of the third one and of the fact that $l_N(B) = 0$ if $N > 0$. □

Eq. (8) shows that this representation of the algebra $\mathcal{A}$ generated by the l_N's is a highest-weight representation with highest weight α. Eq. (6) means that this representation is degenerate at level 2, and it is easy to verify that this implies that the highest weight α is equal to $5/8$ (see [6]). In other words, the fact that one obtains a representation of $\mathcal{A}$ is a consequence of the restriction property, and its degeneracy follows from the Markovian property.

5. Generalizations

Analogous arguments can be used to relate other chordal SLE's to degenerate highest-weight representations of the Virasoro algebra (which is the central extension of $\mathcal{A}$). When $\kappa \neq 8/3$, SLE does not satisfy the conformal restriction property. However, it is possible to quantify how much the property fails to be true and this [19] involves the integral of a certain constant $c(\kappa)$ times the Schwarzian derivative of some conformal maps (see also [19,20] for another interpretation). This constant $c(\kappa)$ will turn out to be exactly the central charge of the corresponding representation. The highest-weight is the exponent of the restriction measure that is naturally associated to the SLE via the correspondance derived in [19]. This will be detailed in a forthcoming paper.

One can also make a similar analysis for the SLE behaviour in the bulk, and make the link to Ward-type identities and representation theory. However, in this case, there are some additional questions to solve if one looks for a completely rigorous link. For instance, the very definition of the "correlation function" (see [2] to see the difficulty of estimating precisely the probability that the SLE paths pass in the neighbourhood of n given points in the upper half-plane) is problematic.

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